Multiple Modalities

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Abstract

Linear logic removes the structural rules of Weakening and Contraction and adds an S4-like modality (written !). Only formulae of the form $!\phi$ can be weakened or contracted. An interesting question is whether these two capabilities can be separated, using two different modalities. This question was studied semantically in a comprehensive paper by Jacobs. This paper considers the question proof-theoretically, giving sequent calculus, natural deduction and axiomatic formulations.

1 Introduction

Girard's linear logic [16] removes the two structural rules of Weakening and Contraction, i.e.[1]

$$\frac{\Gamma \vdash \psi}{\Gamma, \phi \vdash \psi}\text{Weakening} \qquad \frac{\Gamma, \phi, \phi \vdash \psi}{\Gamma, \phi \vdash \psi}\text{Contraction}$$

This has the profound effect of dividing the connectives into two variants, additive and multiplicative. (For the most part of this paper, only the (multiplicative) implication connective is considered.) Unfortunately, a logic without these two structural rules is very weak. Linear logic restores their effect, but in a controlled way. An S4-like modality, !, is introduced with the usual rules

$$\frac{\Gamma, \phi \vdash \psi}{\Gamma, !\phi \vdash \psi}!_{\mathcal{L}} \qquad \text{and} \qquad \frac{!\Gamma \vdash \phi}{!\Gamma \vdash !\phi}!_{\mathcal{R}}$$

where $!\Gamma$ means that all formulae in the multiset Γ are of the form $!\psi_i$. The structural rules are then permitted only on formulae of the form $!\phi$, i.e.

$$\frac{\Gamma \vdash \psi}{\Gamma, !\phi \vdash \psi}\text{Weakening} \qquad \frac{\Gamma, !\phi, !\phi \vdash \psi}{\Gamma, !\phi \vdash \psi}\text{Contraction}$$

However, there is no *a priori* reason why these two structural rules should be associated with the same modality. Jacobs [20] has considered the case where Weakening is associated with one modality, written $!^w$, and Contraction is associated with another, different modality, written $!^c$. However his treatment is, by his own admission, purely semantic. In this paper I wish to consider the proof theoretic implications of having more than one modality. Thus this paper should really be considered as an (hopefully interesting!) addendum to Jacobs' work.

2 Sequent Calculus

2.1 Weakening and Contraction

This section considers the logic studied by Jacobs [20]. As mentioned earlier, there are two new modalities, $!^w$ and $!^c$, which allow formulae to be weakened and contracted, respectively. Of course, one could study the logic where there are these two separate modalities, with no interaction between them, but that seems a little uninteresting. Jacobs considers examples where the familiar modality, !, can be recovered semantically as a particular combination of the two new modalities. Consequently I shall consider the case where there is another modality, written !, where a formula $!\phi$ can be either weakened or contracted.

Thus formulae are given by the grammar

$$\phi ::= p \mid \phi \multimap \phi \mid !^w\phi \mid !^c\phi \mid !\phi$$

(where p is taken from a given set of atomic formulae). I shall refer to a formula of the form $!\phi$, $!^w\phi$ or $!^c\phi$ as a modal formula. There is an obvious ordering on the modalities as follows.

[1] The majority of this paper will consider only the intuitionistic fragment of linear logic (**ILL**). The classical fragment is briefly discussed in §6.

$$\begin{tikzcd} & ! \arrow[dl, dash] \arrow[dr, dash] & \\ !^w & & !^c \end{tikzcd}$$

Sequents are of the form $\Gamma \vdash \phi$, where Γ denotes a multiset of formulae. The sequent calculus formulation is as follows.

$$\phi \vdash \phi$$

$$\frac{\Gamma \vdash \phi \qquad \Delta, \phi \vdash \psi}{\Gamma, \Delta \vdash \psi}\ \mathrm{Cut}$$

$$\frac{\Gamma \vdash \phi \quad \Delta, \psi \vdash \varphi}{\Gamma, \phi \multimap \psi, \Delta \vdash \varphi}\ {\multimap}_{\mathcal{L}} \qquad \frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \multimap \psi}\ {\multimap}_{\mathcal{R}}$$

$$\frac{\Gamma, \phi \vdash \psi}{\Gamma, !^w\phi \vdash \psi}\ !^w_{\mathcal{L}} \qquad \frac{!^w\Gamma, !\Delta \vdash \psi}{!^w\Gamma, !\Delta \vdash !^w\psi}\ !^w_{\mathcal{R}}$$

$$\frac{\Gamma, \phi \vdash \psi}{\Gamma, !^c\phi \vdash \psi}\ !^c_{\mathcal{L}} \qquad \frac{!^c\Gamma, !\Delta \vdash \psi}{!^c\Gamma, !\Delta \vdash !^c\psi}\ !^c_{\mathcal{R}}$$

$$\frac{\Gamma, \phi \vdash \psi}{\Gamma, !\phi \vdash \psi}\ !_{\mathcal{L}} \qquad \frac{!\Delta \vdash \psi}{!\Delta \vdash !\psi}\ !_{\mathcal{R}}$$

$$\frac{\Gamma \vdash \psi}{\Gamma, !^w\phi \vdash \psi}\ \mathrm{Weakening} \qquad \frac{\Gamma \vdash \psi}{\Gamma, !\phi \vdash \psi}\ \mathrm{Weakening}^!$$

$$\frac{\Gamma, !^c\phi, !^c\phi \vdash \psi}{\Gamma, !^c\phi \vdash \psi}\ \mathrm{Contraction} \qquad \frac{\Gamma, !\phi, !\phi \vdash \psi}{\Gamma, !\phi \vdash \psi}\ \mathrm{Contraction}^!$$

The rules concerning the modalities probably need further explanation. All three modalities have the familiar introduction left rule ($!^w_{\mathcal{L}}, !^c_{\mathcal{L}}, !_{\mathcal{L}}$). As discussed above, the motivation is that formulae of the form $!^w\phi$ or $!\phi$ can be weakened ($\mathrm{Weakening}, \mathrm{Weakening}^!$). Similarly formulae of the form $!^c\phi$ or $!\phi$ can be contracted ($\mathrm{Contraction}, \mathrm{Contraction}^!$). The interesting rules are the introduction right rules for the modalities. As mentioned in the introduction, the normal restriction is that when introducing an S4-like modality, $\Box$, on the right, the left hand side formulae must all be of the form $\Box\psi_i$. As we have an ordering on the modalities, this restriction must be changed to the following: to introduce the modality $\Box$ on the right, the left hand formulae must all be of the form $\circ\psi_i$ where $\circ \geq \Box$. This generalisation is discussed further in §2.3.

The important property of this formulation is that it satisfies Gentzen's Hauptsatz.

Theorem 1. *This formulation satisfies the cut-elimination property.*

Proof. The proof is a simple adaptation of the proof for (single-modality) **ILL** [10]. The two interesting cases are the following.

- $$\dfrac{\dfrac{!^w\Gamma, !\Delta \vdash \phi}{!^w\Gamma, !\Delta \vdash !^w\phi}\,!^w_{\mathcal{R}} \qquad \dfrac{\Sigma \vdash \psi}{\Sigma, !^w\phi \vdash \psi}\,\text{Weakening}}{!^w\Gamma, !\Delta, \Sigma \vdash \psi}\,\text{Cut}$$

  This cut is reduced to the following.

  $$\dfrac{\dfrac{\Sigma \vdash \psi}{!^w\Gamma, \Sigma \vdash \psi}\,\text{Weakening}}{!^w\Gamma, !\Delta, \Sigma \vdash \psi}\,\text{Weakening}^!$$

  (where the double lines represent multiple applications of the rule.)

- $$\dfrac{\dfrac{!^c\Gamma, !\Delta \vdash \phi}{!^c\Gamma, !\Delta \vdash !^c\phi}\,!^c_{\mathcal{R}} \qquad \dfrac{\Sigma, !^c\phi, !^c\phi \vdash \psi}{\Sigma, !^c\phi \vdash \psi}\,\text{Contraction}}{!^c\Gamma, !\Delta, \Sigma \vdash \psi}\,\text{Cut}$$

  This cut is reduced to the following.

  $$\dfrac{\dfrac{\dfrac{\dfrac{!^c\Gamma, !\Delta \vdash \phi}{!^c\Gamma, !\Delta \vdash !^c\phi}\,!^c_{\mathcal{R}} \qquad \dfrac{\dfrac{!^c\Gamma, !\Delta \vdash \phi}{!^c\Gamma, !\Delta \vdash !^c\phi}\,!^c_{\mathcal{R}} \qquad \Sigma, !^c\phi, !^c\phi \vdash \psi}{!^c\Gamma, !\Delta, \Sigma, !^c\phi \vdash \psi}\,\text{Cut}}{!^c\Gamma, !^c\Gamma, !\Delta, !\Delta, \Sigma \vdash \psi}\,\text{Cut}}{!^c\Gamma, !\Delta, !\Delta, \Sigma \vdash \psi}\,\text{Contraction}}{!^c\Gamma, !\Delta, \Sigma \vdash \psi}\,\text{Contraction}^!$$

■

One can see the ordering of the modalities in the following derived rules.

Lemma 1. *The following are derived rules.*

$$\dfrac{\Gamma, !^w\phi \vdash \psi}{\Gamma, !\phi \vdash \psi}\,\text{Ordering}_1 \qquad \dfrac{\Gamma, !^c\phi \vdash \psi}{\Gamma, !\phi \vdash \psi}\,\text{Ordering}_2$$

2.2 Linearity, Weakening and Contraction

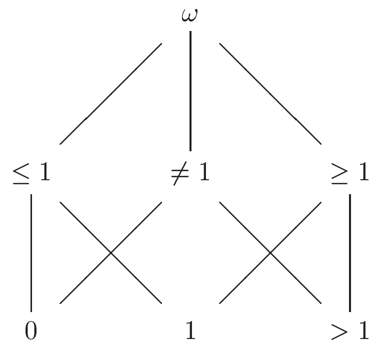
In earlier work [9] I considered a version of **ILL** where there is a family of modalities.[2] The motivation was that proofs in intuitionistic logic (**IL**) would be translated into this logic using a variant of the Girard translation, e.g. for the function type

$$|\sigma \to \tau|^\circ \stackrel{\text{def}}{=} !^i|\sigma|^\circ \multimap |\tau|^\circ$$

where i indicates the *usage* of the argument. From an computational perspective, three kinds of usage are worth distinguishing: where the argument is not used, where it is used exactly

[2]This idea has been re-discovered, in various guises, by a number of people e.g. [2, 15, 27, 22, 28].

once, and where it used many times. An ordering of modalities is then naturally induced and is as follows.[3]



The computational intuition of this translation is summarised by the following table.

the type	is assigned to a function which
$!^{0}\phi \multimap \psi$	uses its argument zero times
$!^{1}\phi \multimap \psi$	uses its argument once
$!^{>1}\phi \multimap \psi$	uses its argument more than once
$!^{\leq 1}\phi \multimap \psi$	uses its argument either once or not at all
$!^{\neq 1}\phi \multimap \psi$	uses its argument either not at all or more than once
$!^{\geq 1}\phi \multimap \psi$	uses its argument either once or many times
$!^{\omega}\phi \multimap \psi$	uses its argument an unknown number of times

Again, using the same methodology of the previous section it is quite straightforward to give a sequent calculus formulation of this logic.

$$\phi \vdash \phi$$

$$\frac{\Gamma \vdash \phi \qquad \Delta, \phi \vdash \psi}{\Gamma, \Delta \vdash \psi}\,\mathrm{Cut}$$

$$\frac{\Gamma \vdash \phi \quad \Delta, \psi \vdash \varphi}{\Gamma, \phi \multimap \psi, \Delta \vdash \varphi}\multimap_{\mathcal{L}} \qquad \frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \multimap \psi}\multimap_{\mathcal{R}}$$

$$\frac{\Gamma, \phi \vdash \psi}{\Gamma, !^{0}\phi \vdash \psi}\,!^{0}_{\mathcal{L}} \qquad \frac{!^{0}\Gamma, !^{\leq 1}\Gamma', !^{\neq 1}\Gamma'', !^{\omega}\Gamma''' \vdash \phi}{!^{0}\Gamma, !^{\leq 1}\Gamma', !^{\neq 1}\Gamma'', !^{\omega}\Gamma''' \vdash !^{0}\phi}\,!^{0}_{\mathcal{R}}$$

$$\frac{\Gamma, \phi \vdash \psi}{\Gamma, !^{1}\phi \vdash \psi}\,!^{1}_{\mathcal{L}} \qquad \frac{!^{1}\Gamma, !^{\leq 1}\Gamma', !^{\geq 1}\Gamma'', !^{\omega}\Gamma''' \vdash \phi}{!^{1}\Gamma, !^{\leq 1}\Gamma', !^{\geq 1}\Gamma'', !^{\omega}\Gamma''' \vdash !^{1}\phi}\,!^{1}_{\mathcal{R}}$$

$$\frac{\Gamma, \phi \vdash \psi}{\Gamma, !^{>1}\phi \vdash \psi}\,!^{>1}_{\mathcal{L}} \qquad \frac{!^{>1}\Gamma, !^{\neq 1}\Gamma', !^{\geq 1}\Gamma'', !^{\omega}\Gamma''' \vdash \phi}{!^{>1}\Gamma, !^{\neq 1}\Gamma', !^{\geq 1}\Gamma'', !^{\omega}\Gamma''' \vdash !^{>1}\phi}\,!^{>1}_{\mathcal{R}}$$

[3]In earlier work [9] an *inference* system based on this type system was considered. In this setup the usage information (modalities) is determined by the solution of a set of constraints. In the case where recursion is present, one possibility is to find *fixed points* of constraint equations. Thus we would naturally add a least element, $\perp$, to the ordering.

$$\frac{\Gamma, \phi \vdash \psi}{\Gamma, !^{\leq 1}\phi \vdash \psi}\,!^{\leq 1}_{\mathcal{L}} \qquad \frac{!^{\leq 1}\Gamma, !^{\omega}\Gamma' \vdash \phi}{!^{\leq 1}\Gamma, !^{\omega}\Gamma' \vdash !^{\leq 1}\phi}\,!^{\leq 1}_{\mathcal{R}}$$

$$\frac{\Gamma, \phi \vdash \psi}{\Gamma, !^{\neq 1}\phi \vdash \psi}\,!^{\neq 1}_{\mathcal{L}} \qquad \frac{!^{\neq 1}\Gamma, !^{\omega}\Gamma' \vdash \phi}{!^{\neq 1}\Gamma, !^{\omega}\Gamma' \vdash !^{\neq 1}\phi}\,!^{\neq 1}_{\mathcal{R}}$$

$$\frac{\Gamma, \phi \vdash \psi}{\Gamma, !^{\geq 1}\phi \vdash \psi}\,!^{\geq 1}_{\mathcal{L}} \qquad \frac{!^{\geq 1}\Gamma, !^{\omega}\Gamma' \vdash \phi}{!^{\geq 1}\Gamma, !^{\omega}\Gamma' \vdash !^{\geq 1}\phi}\,!^{\geq 1}_{\mathcal{R}}$$

$$\frac{\Gamma, \phi \vdash \psi}{\Gamma, !^{\omega}\phi \vdash \psi}\,!^{\omega}_{\mathcal{L}} \qquad \frac{!^{\omega}\Gamma \vdash \phi}{!^{\omega}\Gamma \vdash !^{\omega}\phi}\,!^{\omega}_{\mathcal{R}}$$

$$\frac{\Gamma \vdash \psi}{\Gamma, !^{0}\phi \vdash \psi}\,\text{Weakening} \qquad \frac{\Gamma \vdash \psi}{\Gamma, !^{\leq 1}\phi \vdash \psi}\,\text{Weakening}^{\leq 1}$$

$$\frac{\Gamma \vdash \psi}{\Gamma, !^{\neq 1}\phi \vdash \psi}\,\text{Weakening}^{\neq 1} \qquad \frac{\Gamma \vdash \psi}{\Gamma, !^{\omega}\phi \vdash \psi}\,\text{Weakening}^{\omega}$$

$$\frac{\Gamma, !^{>1}\phi, !^{>1}\phi \vdash \psi}{\Gamma, !^{>1}\phi \vdash \psi}\,\text{Contraction} \qquad \frac{\Gamma, !^{\neq 1}\phi, !^{\neq 1}\phi \vdash \psi}{\Gamma, !^{\neq 1}\phi \vdash \psi}\,\text{Contraction}^{\neq 1}$$

$$\frac{\Gamma, !^{\geq 1}\phi, !^{\geq 1}\phi \vdash \psi}{\Gamma, !^{\geq 1}\phi \vdash \psi}\,\text{Contraction}^{\geq 1} \qquad \frac{\Gamma, !^{\omega}\phi, !^{\omega}\phi \vdash \psi}{\Gamma, !^{\omega}\phi \vdash \psi}\,\text{Contraction}^{\omega}$$

Theorem 2. *This formulation satisfies the cut-elimination property.*

The ordering of the modalities is reflected by the fact that the following are derived rules.

$$\frac{\Gamma, !^{i}\phi \vdash \psi}{\Gamma, !^{j}\phi \vdash \psi}\,\text{Ordering}_1 \quad (j > i) \qquad \frac{\Gamma \vdash !^{j}\phi}{\Gamma \vdash !^{i}\phi}\,\text{Ordering}_2 \quad (i \leq j)$$

2.3 Generalised System

The reader will have noticed that, to some extent, much of the work in giving the sequent calculus formulations in the previous subsections was quite automatic. This can be formalised as follows.

Definition 1. *A* multi-modality **ILL**, $(P, \multimap, M, \leq)$, *consists of*

P	*a set of atomic formulae*
$M = \{!^{i}\}$	*a family of modalities*
$\leq \subseteq M \times M$	*an ordering on the modalities.*

Lemma 2. *A sequent calculus formulation of a multi-modality* **ILL**, $(P, \multimap, M, \leq)$, *consists of the rules*

$$\frac{}{\phi \vdash \phi}\ \text{Identity}$$

$$\frac{\Gamma \vdash \phi \qquad \Delta, \phi \vdash \psi}{\Gamma, \Delta \vdash \psi}\ \text{Cut}$$

$$\frac{\Gamma \vdash \phi \quad \Delta, \psi \vdash \varphi}{\Gamma, \phi \multimap \psi, \Delta \vdash \varphi} \multimap_{\mathcal{L}} \quad \frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \multimap \psi} \multimap_{\mathcal{R}}$$

and $\forall 1 \leq i \leq |M|$

$$\frac{\Gamma, \phi \vdash \psi}{\Gamma, !^i \phi \vdash \psi}\ !^i_{\mathcal{L}}$$

$$\frac{\Gamma \vdash \psi}{\Gamma \vdash !^i \psi}\ !^i_{\mathcal{R}} \qquad \textit{where } \forall 1 \leq j \leq |\Gamma|.\ \Gamma_j \equiv !^k \phi \textit{ and } k \geq i$$

Of course, this will only yield a logic with left and right introduction rules for the modalities. If we wish to add capabilities (extra rules) to modal formulae (e.g. allow formulae of the form $!^w \phi$ to be weakened), then one should ensure that this capability is inherited by the modalities which are greater in the modality ordering. This can be seen quite clearly in the formulations of §§2.1–2. For example, in §2.1 the 'capability' of the $!^w$ modality is the Weakening rule—this is inherited by the $!(\geq !^w)$ modality, yielding the Weakening$^!$ rule. Of course, there is no guarantee that this procedure will yield a well behaved proof theory—one can propose any number of bizarre rules! The point of this section is to demonstrate that if one does have a well behaved logic with multiple modalities (presumably verified by some form of model theory), then there is a simple method for deriving a sequent calculus formulation.

3 Natural Deduction

3.1 Modalities

The problems with presenting S4-like modalities in natural deduction are discussed by Prawitz [24] and re-explored by Bierman and de Paiva [12]. The elimination rule is quite straightforward, i.e.

$$\frac{\begin{array}{c}\vdots \\ \Box\phi\end{array}}{\phi}\ \Box_{\mathcal{E}}$$

The problem lies in giving an introduction rule. Obviously we have to ensure that the open assumptions are of the form $\Box\phi_i$. A first attempt at the introduction rule would be

$$\frac{\begin{array}{c}\Box\phi_1 \cdots \Box\phi_k \\ \vdots \\ \psi\end{array}}{\Box\psi}\ \Box_{\mathcal{I}},$$

where the assumptions must all be modal. The problem is that this rule is clearly *not* closed under substitution. For example, substituting for $\Box\phi_1$, the deduction

$$\frac{\varphi \wedge \Box\phi_1}{\Box\phi_1}\wedge_{\mathcal{E}}$$

we get

$$\cfrac{\begin{array}{c}\dfrac{\varphi \wedge \Box\phi_1}{\Box\phi_1}\wedge_{\mathcal{E}} \quad \cdots \Box\phi_k \\ \vdots \\ \psi\end{array}}{\Box\psi}\Box_{\mathcal{I}}$$

which is no longer a valid deduction, as not all the open assumptions are modal.

In his monograph [24, Chapter VI] Prawitz suggests a notion of "essentially modal" formulae. What this amounts to is a relaxing of the restriction that all the undischarged formulae are modal, but rather that there is somewhere in the deduction a *complete set* of modal formulae which could have had deductions substituted in for them. In tree-form this amounts to the rule (where the complete set of formulae is $\Box\phi_1 \cdots \Box\phi_k$)

$$\cfrac{\begin{array}{c}\begin{array}{ccc}\Delta_1 & & \Delta_k \\ \vdots & & \vdots \\ \Box\phi_1 & \cdots & \Box\phi_k\end{array} \\ \vdots \\ \psi\end{array}}{\Box\psi}\Box_{\mathcal{I}}.$$

but, as Bierman and de Paiva show [12], this forces an isomorphism $\Box\phi \cong \Box\Box\phi$ (which is not true in many models of interest). A solution is to not only insist that all open assumptions are modal, but to immediately discharge and re-introduce them, i.e.

$$\cfrac{\begin{array}{c}\vdots \\ \Box\phi_1\end{array} \;\; \ldots \;\; \begin{array}{c}\vdots \\ \Box\phi_k\end{array} \qquad \begin{array}{c}[\![\Box\phi_1 \cdots \Box\phi_k]\!] \\ \vdots \\ \psi\end{array}}{\Box\psi}\Box_{\mathcal{I}}$$

(The semantic braces, $[\![\cdots]\!]$, serve to remind that *all* the assumptions must be modal and discharged.) It is easy to see that this rule satisfies the property of closure under substitution.

This formulation was first used for the ! modality of **ILL** by Benton *et al.* [7] and for the necessity modality of the modal logic **IS4**, by Bierman and de Paiva [12]. Subsequently, a number of other presentations of the modality have been proposed for **ILL**; most notably Barber's dual context formulation [3] and Benton's dual system formulation [5]. All three presentations will be considered, although for brevity I shall take only the three modality logic described in §2.1.

3.2 Single context formulation

For compactness, deductions are written in 'sequent-style', i.e.

$$\Gamma \vdash \phi$$

where Γ is the multiset of open assumptions (and is often referred to as the context). The formulation is as follows.

$$\phi \vdash \phi$$

$$\frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \multimap \psi} \multimap_{\mathcal{I}} \qquad \frac{\Gamma \vdash \phi \multimap \psi \qquad \Delta \vdash \phi}{\Gamma, \Delta \vdash \psi} \multimap_{\mathcal{E}}$$

$$\frac{\Gamma_1 \vdash !^w\psi_1 \quad \cdots \quad \Gamma_k \vdash !^w\psi_k \quad \Delta_1 \vdash !\varphi_1 \quad \cdots \quad \Delta_l \vdash !\varphi_l \qquad !^w\psi_1, \ldots, !^w\psi_k, !\varphi_1, \ldots, !\varphi_l \vdash \phi}{\Gamma_1, \ldots, \Gamma_k, \Delta_1, \ldots, \Delta_l \vdash !^w\phi} !^w_{\mathcal{I}}$$

$$\frac{\Gamma \vdash !^w\phi}{\Gamma \vdash \phi} !^w_{\mathcal{E}}$$

$$\frac{\Gamma_1 \vdash !^c\psi_1 \quad \cdots \quad \Gamma_k \vdash !^c\psi_k \quad \Delta_1 \vdash !\varphi_1 \quad \cdots \quad \Delta_l \vdash !\varphi_l \qquad !^c\psi_1, \ldots, !^c\psi_k, !\varphi_1, \ldots, !\varphi_l \vdash \phi}{\Gamma_1, \ldots, \Gamma_k, \Delta_1, \ldots, \Delta_l \vdash !^c\phi} !^c_{\mathcal{I}}$$

$$\frac{\Gamma \vdash !^c\phi}{\Gamma \vdash \phi} !^c_{\mathcal{E}}$$

$$\frac{\Gamma_1 \vdash !\psi_1 \quad \cdots \quad \Gamma_k \vdash !\psi_k \qquad !\psi_1, \ldots, !\psi_k \vdash \phi}{\Gamma_1, \ldots, \Gamma_k \vdash !\phi} !_{\mathcal{I}}$$

$$\frac{\Gamma \vdash !\phi}{\Gamma \vdash \phi} !_{\mathcal{E}}$$

$$\frac{\Gamma \vdash !^w\phi \qquad \Delta \vdash \psi}{\Gamma, \Delta \vdash \psi} \text{Weakening} \qquad \frac{\Gamma \vdash !\phi \qquad \Delta \vdash \psi}{\Gamma, \Delta \vdash \psi} \text{Weakening}^!$$

$$\frac{\Gamma \vdash !^c\phi \qquad \Delta, !^c\phi, !^c\phi \vdash \psi}{\Gamma, \Delta \vdash \psi} \text{Contraction} \qquad \frac{\Gamma \vdash !\phi \qquad \Delta, !\phi, !\phi \vdash \psi}{\Gamma, \Delta \vdash \psi} \text{Contraction}^!$$

The modality ordering is preserved in this formulation, as given by the following lemma.

Lemma 3. *The following rules are admissible.*

$$\frac{\Gamma, !^w\phi \vdash \psi}{\Gamma, !\phi \vdash \psi} \text{Order}_1 \qquad \frac{\Gamma \vdash !\phi}{\Gamma \vdash !^w\phi} \text{Order}_2$$

$$\frac{\Gamma, !^c\phi \vdash \psi}{\Gamma, !\phi \vdash \psi} \text{Order}_3 \qquad \frac{\Gamma \vdash !\phi}{\Gamma \vdash !^c\phi} \text{Order}_4$$

As normal a substitution rule is admissible, i.e.

$$\frac{\Gamma \vdash \phi \qquad \Delta, \phi \vdash \psi}{\Gamma, \Delta \vdash \psi} \text{Substitution}$$

It is easy to see that this formulation is equivalent to the sequent calculus formulation given in §2.1, in that deductions can be mapped from one formulation to another. For example, consider the following instance of the sequent calculus rule $!^w_{\mathcal{R}}$.

$$\frac{!^w\phi, !\psi \vdash \varphi}{!^w\phi, !\psi \vdash \varphi}\, !^w_{\mathcal{R}}$$

Assuming that the upper sequent is mapped to a deduction $\mathcal{D}$, the instance of the rule can be mapped to the following natural deduction.

$$\frac{\overline{!^w\phi \vdash !^w\phi} \qquad \overline{!\psi \vdash !\psi} \qquad \overset{\mathcal{D}}{!^w\phi, !\psi \vdash \varphi}}{!^w\phi, !\psi \vdash !^w\varphi}\, !^w_{\mathcal{I}}$$

Following this line of argument it is quite simple to show that the two formulations are equivalent (where a deduction in the sequent calculus formulation is prefixed with $\vdash^S$ and in the natural deduction formulation $\vdash^N$).

Theorem 3. $\vdash^N \Gamma \vdash \phi$ *iff* $\vdash^S \Gamma \vdash \phi$.

3.3 Multi-context formulation

Recall the problems discussed in §3.1: one of the difficulties with the introduction rule was ensuring the condition that all the open assumptions (the context) were modal. An alternative would be to split the context into two, where one part contains only modal formulae and the other the non-modal formulae. The condition then becomes a check that the non-modal context is empty. More precisely, judgements are of the form

$$\Gamma; \Delta \vdash \phi$$

where Γ is the modal context and Δ the non-modal context. To reiterate: the idea is that this judgement corresponds to the judgement

$$\Box\Gamma, \Delta \vdash \phi$$

of the system described in §3.1.

This 'multi-context' approach was studied comprehensively in the context of **ILL** by Barber [3] and in the context of cut-free proof search by Hodas and Miller [19]. To handle the multiple modalities, judgements are now of the form

$$\Gamma; \Delta; \Sigma; \Theta \vdash \phi$$

where Γ is the $!^w$ context (multiset), Δ the $!^c$ context (set), Σ the $!$ context (set), and Θ the linear context (multiset). In other words, it can be thought of as representing the judgement

$$!^w\Gamma, !^c\Delta, !\Sigma, \Theta \vdash \phi$$

of the system described in §3.2.

The formulation is then as follows.

$$\frac{}{\Gamma; -; \Sigma; \phi \vdash \phi}\ \text{Identity} \qquad \frac{}{\Gamma, \phi; -; \Sigma; - \vdash \phi}\ \text{Identity}^w$$

$$\frac{}{\Gamma; \phi; \Sigma; - \vdash \phi}\ \text{Identity}^c \qquad \frac{}{\Gamma; -; \Sigma, \phi; - \vdash \phi}\ \text{Identity}^!$$

$$\frac{\Gamma; \Delta; \Sigma; \Theta, \phi \vdash \psi}{\Gamma; \Delta; \Sigma; \Theta \vdash \phi \multimap \psi}\ {\multimap}_{\mathcal{I}}$$

$$\frac{\Gamma; \Delta, \Delta'; \Sigma; \Theta \vdash \phi \multimap \psi \qquad \Gamma'; \Delta, \Delta''; \Sigma; \Theta' \vdash \phi}{\Gamma, \Gamma'; \Delta, \Delta', \Delta''; \Sigma; \Theta, \Theta' \vdash \psi}\ {\multimap}_{\mathcal{E}}$$

$$\frac{\Gamma; -; \Sigma; - \vdash \phi}{\Gamma; -; \Sigma; - \vdash !^w \phi}\ !^w_{\mathcal{I}} \qquad \frac{\Gamma; \Delta, \Delta'; \Sigma; \Theta \vdash !^w \phi \qquad \Gamma', \phi; \Delta, \Delta''; \Sigma; \Theta' \vdash \psi}{\Gamma, \Gamma'; \Delta, \Delta', \Delta''; \Sigma; \Theta, \Theta' \vdash \psi}\ !^w_{\mathcal{E}}$$

$$\frac{-; \Delta; \Sigma; - \vdash \phi}{\Gamma; \Delta; \Sigma; - \vdash !^c \phi}\ !^c_{\mathcal{I}} \qquad \frac{\Gamma; \Delta, \Delta'; \Sigma; \Theta \vdash !^c \phi \qquad \Gamma'; \Delta, \phi, \Delta''; \Sigma; \Theta' \vdash \psi}{\Gamma, \Gamma'; \Delta, \Delta', \Delta''; \Sigma; \Theta, \Theta' \vdash \psi}\ !^c_{\mathcal{E}}$$

$$\frac{-; -; \Sigma; - \vdash \phi}{\Gamma; -; \Sigma; - \vdash ! \phi}\ !_{\mathcal{I}} \qquad \frac{\Gamma; \Delta, \Delta'; \Sigma; \Theta \vdash ! \phi \qquad \Gamma'; \Delta, \Delta''; \Sigma, \phi; \Theta' \vdash \psi}{\Gamma, \Gamma'; \Delta, \Delta', \Delta''; \Sigma; \Theta, \Theta' \vdash \psi}\ !_{\mathcal{E}}$$

It is worth making a few observations about this formulation. Firstly, the Identity rules build-in the Weakening rule for the $!^w$ and $!$ contexts. Secondly, consider an instance of the Identity^c rule

$$-; \phi; -; - \vdash \phi.$$

Using the intuition above, it represents the judgement

$$!^c \phi \vdash \phi.$$

Thus the Identity$^{w,c,!}$ rules all have an implicit modality elimination action. Secondly the handling of the $!^c$ contexts needs some explanation. Consider the ${\multimap}_{\mathcal{E}}$ rule. The idea is that the formulae which are considered common to both upper deductions (Δ) are implicitly contracted in the lower deduction. Thus in the lower deduction Δ, Δ' and Δ'' must be disjoint sets of formulae. This allows the contraction rule to be admissible in the $!^c$ context.[4] Finally, the reader should note that the $!^c_{\mathcal{I}}$ and $!_{\mathcal{I}}$ rules both have an implicit weakening action.

This formulation admits a number of admissible rules.

Lemma 4. *The following rules are admissible.*

$$\frac{\Gamma; \Delta; \Sigma; \Theta \vdash \phi}{\Gamma, \psi; \Delta; \Sigma; \Theta \vdash \phi}\ \text{Weakening}^w \qquad \frac{\Gamma; \Delta; \Sigma; \Theta \vdash \phi}{\Gamma; \Delta; \Sigma, \psi; \Theta \vdash \phi}\ \text{Weakening}^!$$

$$\frac{\Gamma; \Delta, \psi, \psi; \Sigma; \Theta \vdash \phi}{\Gamma; \Delta, \psi; \Sigma; \Theta \vdash \phi}\ \text{Contraction}^c \qquad \frac{\Gamma; \Delta; \Sigma, \psi, \psi; \Theta \vdash \phi}{\Gamma; \Delta; \Sigma, \psi; \Theta \vdash \phi}\ \text{Contraction}^!$$

[4] An alternative formulation would be to make the $!^c$ a multiset and provide an explicit contraction rule.

$$\frac{\Gamma; \Delta; \Sigma; \Theta \Vdash \phi}{\Gamma; \Delta; \Sigma; \Theta, !^w\psi \Vdash \phi}\,\text{Weakening}^l \qquad \frac{\Gamma; \Delta; \Sigma; \Theta, !^c\psi, !^c\psi \Vdash \phi}{\Gamma; \Delta; \Sigma; \Theta, !^c\psi \Vdash \phi}\,\text{Contraction}^l$$

$$\frac{\Gamma; \Delta, \Delta'; \Sigma; \Theta \Vdash \phi \qquad \Gamma'; \Delta, \Delta''; \Sigma; \Theta', \phi \Vdash \psi}{\Gamma, \Gamma'; \Delta, \Delta', \Delta''; \Sigma; \Theta, \Theta' \Vdash \psi}\,\text{Substitution}$$

The modality ordering is reflected in the following admissible rules.

Lemma 5. *The following rules are admissible.*

$$\frac{\Gamma, \psi; \Delta; \Sigma; \Theta \Vdash \phi}{\Gamma; \Delta; \Sigma, \psi; \Theta \Vdash \phi}\,\text{Order}^1_{\mathcal{L}} \qquad \frac{\Gamma; \Delta, \psi; \Sigma; \Theta \Vdash \phi}{\Gamma; \Delta; \Sigma, \psi; \Theta \Vdash \phi}\,\text{Order}^2_{\mathcal{L}}$$

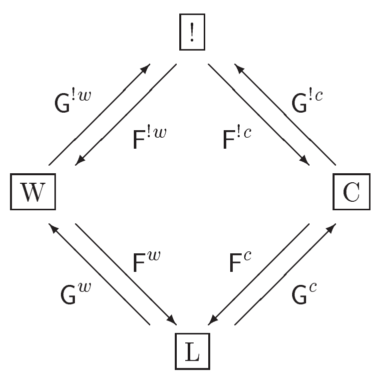
$$\frac{\Gamma; \Delta; \Sigma; \Theta, \phi \Vdash \psi}{\Gamma, \phi; \Delta; \Sigma; \Theta \Vdash \psi}\,\text{Order}^3_{\mathcal{L}} \qquad \frac{\Gamma; \Delta; \Sigma; \Theta, \phi \Vdash \psi}{\Gamma; \Delta, \phi; \Sigma; \Theta \Vdash \psi}\,\text{Order}^4_{\mathcal{L}}$$

Two other interesting derived rules are the following.

$$\frac{\Gamma; \Delta; \Sigma; \Theta \Vdash !\phi}{\Gamma; \Delta; \Sigma; \Theta \Vdash !^w\phi}\,\text{Order}^1_{\mathcal{R}} \qquad \frac{\Gamma; \Delta; \Sigma; \Theta \Vdash !\phi}{\Gamma; \Delta; \Sigma; \Theta \Vdash !^c\phi}\,\text{Order}^2_{\mathcal{R}}$$

3.4 Multi-system Formulation

Another intriguing presentation of modalities was given in the setting of **ILL** by Benton [5]. In the multi-context formulation given in the previous section, the idea is to have different classes of formulae, but only one class of deduction. In the multi-system formulation we allow different classes, or worlds, of deduction as well. Modalities are then decomposed into operations which move formulae between one world and another. Our three modality logic yields four worlds which are connected as follows.



The form of a formula determines which world it lives in. Formulae are given by the following mutually recursive grammars (where p is again taken from a given set of atomic formulae).

$$\begin{array}{lccl}
\text{l}-\text{formulae} & \theta & ::= & p \mid \theta \multimap \theta \mid \mathsf{F}^w(\gamma) \mid \mathsf{F}^c(\delta) \\
\text{w}-\text{formulae} & \gamma & ::= & \mathsf{G}^w(\theta) \mid \mathsf{F}^{!w}(\sigma) \\
\text{c}-\text{formulae} & \delta & ::= & \mathsf{G}^c(\theta) \mid \mathsf{F}^{!c}(\sigma) \\
!-\text{formulae} & \sigma & ::= & \mathsf{G}^{!w}(\gamma) \mid \mathsf{G}^{!c}(\delta)
\end{array}$$

I shall use the appropriate capital Greek letter to denote a multiset of formulae, e.g. Δ denotes a multiset of c-formulae. Corresponding to the four worlds there are four forms of deductions, which are as follows.

$$\begin{tikzcd}
& \Sigma \vdash_{!} \sigma \arrow[dl, dotted, shift left] \arrow[dr, dotted, shift left] & \\
\Gamma; \Sigma \vdash_{w} \gamma \arrow[ur, dotted, shift left] \arrow[dr, dotted, shift left] & & \Delta; \Sigma \vdash_{c} \delta \arrow[ul, dotted, shift left] \arrow[dl, dotted, shift left] \\
& \Gamma; \Delta; \Sigma; \Theta \vdash_{l} \theta \arrow[ul, dotted, shift left] \arrow[ur, dotted, shift left] &
\end{tikzcd}$$

The multi-system formulation is then as follows.

$$\frac{}{\Gamma; -; \Sigma; \theta \vdash_{l} \theta}\ \mathrm{Identity}^{l} \qquad \frac{}{\Gamma, \gamma; \Sigma \vdash_{w} \gamma}\ \mathrm{Identity}^{w}$$

$$\frac{}{\delta; \Sigma \vdash_{c} \delta}\ \mathrm{Identity}^{c} \qquad \frac{}{\Sigma, \sigma \vdash_{!} \sigma}\ \mathrm{Identity}^{!}$$

$$\frac{\Gamma; \Delta; \Sigma; \Theta, \theta \vdash_{l} \theta'}{\Gamma; \Delta; \Sigma; \Theta \vdash_{l} \theta \multimap \theta'}\ {\multimap_{\mathcal{I}}} \qquad \frac{\Gamma; \Delta, \Delta'; \Sigma; \Theta \vdash_{l} \theta \multimap \theta' \quad \Gamma'; \Delta, \Delta''; \Sigma; \Theta' \vdash_{l} \theta}{\Gamma, \Gamma'; \Delta, \Delta', \Delta''; \Sigma; \Theta, \Theta' \vdash_{l} \theta'}\ {\multimap_{\mathcal{E}}}$$

$$\frac{\Gamma; \Sigma \vdash_{w} \gamma}{\Gamma; -; \Sigma; - \vdash_{l} \mathsf{F}^{w}(\gamma)}\ \mathsf{F}^{w}_{\mathcal{I}} \qquad \frac{\Gamma; \Delta, \Delta'; \Sigma; \Theta \vdash_{l} \mathsf{F}^{w}(\gamma) \quad \Gamma', \gamma; \Delta, \Delta''; \Sigma; \Theta' \vdash_{l} \theta}{\Gamma, \Gamma'; \Delta, \Delta', \Delta''; \Sigma; \Theta, \Theta' \vdash_{l} \theta}\ \mathsf{F}^{w}_{\mathcal{E}}$$

$$\frac{\Delta; \Sigma \vdash_{c} \delta}{\Gamma; \Delta; \Sigma; - \vdash_{l} \mathsf{F}^{c}(\delta)}\ \mathsf{F}^{c}_{\mathcal{I}} \qquad \frac{\Gamma; \Delta, \Delta'; \Sigma; \Theta \vdash_{l} \mathsf{F}^{c}(\delta) \quad \Gamma'; \Delta, \delta, \Delta''; \Sigma; \Theta' \vdash_{l} \theta}{\Gamma, \Gamma'; \Delta, \Delta', \Delta''; \Sigma; \Theta, \Theta' \vdash_{l} \theta}\ \mathsf{F}^{c}_{\mathcal{E}}$$

$$\frac{\Sigma \vdash_{!} \sigma}{\Gamma; \Sigma \vdash_{w} \mathsf{F}^{!w}(\sigma)}\ \mathsf{F}^{!w}_{\mathcal{I}} \qquad \frac{\Gamma; \Sigma \vdash_{w} \mathsf{F}^{!w}(\sigma) \quad \Gamma'; \Sigma, \sigma \vdash_{w} \gamma}{\Gamma, \Gamma'; \Sigma \vdash_{w} \gamma}\ \mathsf{F}^{!w}_{\mathcal{E}}$$

$$\frac{\Sigma \vdash_{!} \sigma}{-; \Sigma \vdash_{c} \mathsf{F}^{!c}(\sigma)}\ \mathsf{F}^{!c}_{\mathcal{I}} \qquad \frac{\Delta; \Sigma \vdash_{c} \mathsf{F}^{!c}(\sigma) \quad \Delta; \Sigma, \sigma \vdash_{c} \delta}{\Delta; \Sigma \vdash_{c} \delta}\ \mathsf{F}^{!c}_{\mathcal{E}}$$

$$\frac{\Gamma; -; \Sigma; - \vdash_{l} \theta}{\Gamma; \Sigma \vdash_{w} \mathsf{G}^{w}(\theta)}\ \mathsf{G}^{w}_{\mathcal{I}} \qquad \frac{\Gamma; \Sigma \vdash_{w} \mathsf{G}^{w}(\theta)}{\Gamma; -; \Sigma; - \vdash_{l} \theta}\ \mathsf{G}^{w}_{\mathcal{E}}$$

$$\frac{-; \Delta; \Sigma; - \vdash_{l} \theta}{\Delta; \Sigma \vdash_{c} \mathsf{G}^{c}(\theta)}\ \mathsf{G}^{c}_{\mathcal{I}} \qquad \frac{\Delta; \Sigma \vdash_{c} \mathsf{G}^{c}(\theta)}{\Gamma; \Delta; \Sigma; - \vdash_{l} \theta}\ \mathsf{G}^{c}_{\mathcal{E}}$$

$$\frac{-; \Sigma \vdash_{w} \gamma}{\Sigma \vdash_{!} \mathsf{G}^{!w}(\gamma)}\ \mathsf{G}^{!w}_{\mathcal{I}} \qquad \frac{\Sigma \vdash_{!} \mathsf{G}^{!w}(\gamma)}{\Gamma; \Sigma \vdash_{w} \gamma}\ \mathsf{G}^{!w}_{\mathcal{E}}$$

$$\frac{-; \Sigma \vdash_{c} \delta}{\Sigma \vdash_{!} \mathsf{G}^{!c}(\delta)}\ \mathsf{G}^{!c}_{\mathcal{I}} \qquad \frac{\Sigma \vdash_{!} \mathsf{G}^{!c}(\delta)}{-; \Sigma \vdash_{c} \delta}\ \mathsf{G}^{!c}_{\mathcal{E}}$$

Benton's formulation follows closely the categorical construction behind the logic. Briefly, modalities are modelled by certain sorts of comonads. These comonads generally give rise to an adjunction between categories, where the adjoint functors are represented explicitly in the formulation by the F and G operators. The categorically inclined reader is referred to Benton's paper [5] for further details concerning this approach and to Jacobs' paper [20] for details of this setting.

An important property of this formulation is that the substitution rule is admissible, i.e.

$$\frac{\Gamma;\Delta,\Delta';\Sigma;\Theta \vdash_l \theta \qquad \Gamma';\Delta,\Delta'';\Sigma;\Theta',\theta \vdash_l \theta'}{\Gamma,\Gamma';\Delta,\Delta',\Delta'';\Sigma;\Theta,\Theta' \vdash_l \theta'}\ \text{Substitution}$$

As expected the structural rules are admissible in the appropriate context, i.e.

$$\frac{\Gamma;\Delta;\Sigma;\Theta \vdash_l \theta}{\Gamma,\gamma;\Delta;\Sigma;\Theta \vdash_l \theta}\ \text{Weakening}_1 \qquad \frac{\Gamma;\Delta;\Sigma;\Theta \vdash_l \theta}{\Gamma;\Delta;\Sigma,\sigma;\Theta \vdash_l \theta}\ \text{Weakening}_2$$

$$\frac{\Gamma;\Delta,\delta,\delta;\Sigma;\Theta \vdash_l \theta}{\Gamma;\Delta,\delta;\Sigma;\Theta \vdash_l \theta}\ \text{Contraction}_1 \qquad \frac{\Gamma;\Delta;\Sigma,\sigma,\sigma;\Theta \vdash_l \theta}{\Gamma;\Delta;\Sigma,\sigma;\Theta \vdash_l \theta}\ \text{Contraction}_2$$

A more detailed study of this formulation is left for future study. The following list gives some rules which are valid in the formulation.

$$\frac{\Gamma,\gamma;\Delta;\Sigma;\Theta \vdash_l \theta}{\Gamma;\Delta;\Sigma;\Theta,\mathsf{F}^w(\gamma) \vdash_l \theta} \qquad \frac{\Gamma;\Delta;\Sigma;\Theta,\theta \vdash_l \theta'}{\Gamma,\mathsf{G}^w(\theta);\Delta;\Sigma;\Theta \vdash_l \theta'}$$

$$\frac{\Gamma,\gamma;\Delta;\Sigma;\Theta \vdash_l \theta}{\Gamma,\mathsf{G}^w(\mathsf{F}^w(\gamma));\Delta;\Sigma;\Theta \vdash_l \theta} \qquad \frac{\Gamma;\Delta;\Sigma;\Theta,\theta \vdash_l \theta'}{\Gamma;\Delta;\Sigma;\Theta,\mathsf{F}^w(\mathsf{G}^w(\theta)) \vdash_l \theta'}$$

$$\frac{\Gamma;\delta,\Delta;\Sigma;\Theta \vdash_l \theta}{\Gamma;\Delta;\Sigma;\Theta,\mathsf{F}^c(\delta) \vdash_l \theta} \qquad \frac{\Gamma;\mathsf{G}^c(\theta),\Delta;\Sigma;\Theta,\theta \vdash_l \theta'}{\Gamma;\mathsf{G}^c(\theta),\Delta;\Sigma;\Theta \vdash_l \theta'}$$

$$\frac{\Gamma;\mathsf{G}^c(\theta),\Delta;\Sigma;\Theta,\theta \vdash_l \theta'}{\Gamma;\Delta;\Sigma;\Theta,\mathsf{F}^c(\mathsf{G}^c(\theta)) \vdash_l \theta'} \qquad \frac{\Gamma;\mathsf{G}^c(\theta),\Delta;\Sigma;\Theta,\theta,\theta \vdash_l \theta'}{\Gamma;\Delta;\Sigma;\Theta,\mathsf{F}^c(\mathsf{G}^c(\theta)) \vdash_l \theta'}$$

$$\frac{\Gamma,\mathsf{F}^{!w}(\sigma);\Delta;\Sigma;\Theta \vdash_l \theta}{\Gamma;\Delta;\Sigma,\sigma;\Theta \vdash_l \theta} \qquad \frac{\Gamma;\mathsf{F}^{!c}(\sigma),\Delta;\Sigma;\Theta \vdash_l \theta}{\Gamma;\Delta;\Sigma,\sigma;\Theta \vdash_l \theta}$$

4 Axiomatic Formulation

Axiomatic, or Hilbert-style, formulations are probably the more familiar method of presenting modal logics and so, for completeness, I shall give an axiomatic formulation of the three modality logic of §2.1.

The axiomatic formulation consists of the following axiom schemas.

$$\begin{array}{rl}
\mathtt{I}: & \phi \multimap \phi \\
\mathtt{B}: & (\psi \multimap \varphi) \multimap ((\phi \multimap \psi) \multimap (\phi \multimap \varphi)) \\
\mathtt{C}: & (\phi \multimap (\psi \multimap \varphi)) \multimap (\psi \multimap (\phi \multimap \varphi)) \\
\\
\mathtt{W}^c: & (!^c\phi \multimap (!^c\phi \multimap \psi)) \multimap (!^c\phi \multimap \psi) \\
\mathtt{W}^!: & (!\phi \multimap (!\phi \multimap \psi)) \multimap (!\phi \multimap \psi)
\end{array}$$

$$
\begin{aligned}
\mathtt{k}^w &: \quad \phi \multimap (!^w\psi \multimap \phi) \\
\mathtt{k}^! &: \quad \phi \multimap (!\psi \multimap \phi) \\
\\
\mathtt{T}^w &: \quad !^w\phi \multimap \phi \\
\mathtt{4}^w &: \quad !^w\phi \multimap !^w!^w\phi \\
\mathtt{K}^w &: \quad !^w(\phi \multimap \psi) \multimap (!^w\phi \multimap !^w\psi) \\
\\
\mathtt{T}^c &: \quad !^c\phi \multimap \phi \\
\mathtt{4}^c &: \quad !^c\phi \multimap !^c!^c\phi \\
\mathtt{K}^c &: \quad !^c(\phi \multimap \psi) \multimap (!^c\phi \multimap !^c\psi) \\
\\
\mathtt{T}^! &: \quad !\phi \multimap \phi \\
\mathtt{4}^! &: \quad !\phi \multimap !!\phi \\
\mathtt{K}^! &: \quad !(\phi \multimap \psi) \multimap (!\phi \multimap !\psi) \\
\\
\mathtt{order}^w &: \quad !\phi \multimap !^w\phi \\
\mathtt{order}^c &: \quad !\phi \multimap !^c\phi
\end{aligned}
$$

The axiom schemas have been given names which, apart from the last two, should be familiar to the reader well-read in modal logics [14] (T,4,K) and combinatory systems [18] (I,B,C,W,k).[5] In addition to the axiom schemas there are the following rules (cf. [26, §9.1]).

$$
\frac{}{\phi \vdash \phi}\ \text{Identity} \qquad \frac{}{\vdash \phi}\ \text{Axiom } (\phi \text{ an instance of an axiom schema})
$$

$$
\frac{\Gamma \vdash \phi \multimap \psi \qquad \Delta \vdash \phi}{\Gamma, \Delta \vdash \psi}\ \text{Modus Ponens}
$$

$$
\frac{\vdash \phi}{\vdash !^w\phi}\ !^w \qquad \frac{\vdash \phi}{\vdash !^c\phi}\ !^c \qquad \frac{\vdash \phi}{\vdash !\phi}\ !
$$

It is essential to note that in the three modality rules, the context *must* be empty. An important property of this axiomatic formulation is the following, which is often called the 'deduction theorem'.

Theorem 4. *If* $\Gamma, \phi \vdash \psi$ *then* $\Gamma \vdash \phi \multimap \psi$.

Proof. This follows by a simple induction. It is clear that the converse holds by an application of Modus Ponens. ∎

It is relatively easy to see that this formulation is equivalent to the other formulations of this logic. For example, consider the following instance of the sequent calculus $!^w_{\mathcal{R}}$ rule.

$$
\frac{!^w\phi, !\psi \vdash \varphi}{!^w\phi, !\psi \vdash !^w\varphi}\ !^w_{\mathcal{R}}
$$

Assume that we have an axiomatic deduction $\mathcal{D}$ of the upper sequent. We can then construct the following axiomatic deduction (where for compactness, only the names of axioms schemas have been given).

[5] It is unfortunate that there are two K combinators in the literature. I have used a capital to distinguish the modal combinator.

$$
\begin{prooftree}
\AxiomC{$\vdash \mathtt{K}^w$}
\AxiomC{$\vdash \mathtt{B}$}
\AxiomC{$\vdash \mathtt{K}^w$}
\AxiomC{$\mathcal{D}$}
\noLine
\UnaryInfC{$!^w\phi, !\psi \vdash \varphi$}
\doubleLine
\RightLabel{D.T.}
\UnaryInfC{$\vdash !^w\phi \multimap (!\psi \multimap \varphi)$}
\UnaryInfC{$\vdash !^w(!^w\phi \multimap (!\psi \multimap \varphi))$}
\BinaryInfC{$\vdash !^w!^w\phi \multimap !^w(!\psi \multimap \varphi)$}
\BinaryInfC{$\vdash (!^w\phi \multimap !^w!^w\phi) \multimap (!^w\phi \multimap !^w(!\psi \multimap \varphi))$}
\AxiomC{$\vdash \mathtt{4}^w$}
\BinaryInfC{$\vdash !^w\phi \multimap !^w(!\psi \multimap \varphi)$}
\AxiomC{$!^w\phi \vdash !^w\phi$}
\BinaryInfC{$!^w\phi \vdash !^w(!\psi \multimap \varphi)$}
\BinaryInfC{$!^w\phi \vdash !^w!\psi \multimap !^w\varphi$}
\noLine
\UnaryInfC{}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{$\vdash \mathtt{B}$}
\AxiomC{$!^w\phi \vdash !^w!\psi \multimap !^w\varphi$}
\BinaryInfC{$!^w\phi \vdash (!\psi \multimap !^w!\psi) \multimap (!\psi \multimap !^w\varphi)$}
\AxiomC{$\vdash \mathtt{B}$}
\AxiomC{$\vdash \mathtt{order}^w$}
\BinaryInfC{$\vdash (!\psi \multimap !!\psi) \multimap (!\psi \multimap !^w!\psi)$}
\AxiomC{$\vdash \mathtt{4}^!$}
\BinaryInfC{$\vdash !^\psi \multimap !^w!\psi$}
\BinaryInfC{$!^w\phi \vdash !\psi \multimap !^w\varphi$}
\AxiomC{$!\psi \vdash !\psi$}
\BinaryInfC{$!^w\phi, !\psi \vdash !^w\varphi$}
\end{prooftree}
$$

(D.T. denotes an application of the deduction theorem.) All the other sequent calculus rules can be translated in the same (lengthy!) way. Combining this with Theorem 3 gives an equivalence theorem between all three formulations (where a deduction in the axiomatic formulation is prefixed $\vdash^A$).

Theorem 5. (Equivalence) $\vdash^A \Gamma \vdash \phi$ *iff* $\vdash^S \Gamma \vdash \psi$ *iff* $\vdash^N \Gamma \vdash \phi$.

5 Term Calculi

5.1 Typed λ-calculi

The Curry-Howard correspondence relates natural deduction formulations of logics with certain λ-calculi. The prototypical example is the correspondence between the natural deduction formulation of propositional minimal logic and the simply typed λ-calculus. The correspondence between **ILL** and the resulting linear λ-calculus has been studied quite closely [10]. In this section I shall describe the λ-calculus which corresponds to the multi-context natural deduction formulation of the three modality logic of §3.3.[6] Of course, there are λ-calculi corresponding to the single context formulation of §3.2 and the multi-system formulation of §3.4—these are left to the interested reader.

Raw terms are given by the following grammar.

$$
\begin{array}{rcll}
M & ::= & x & \text{Variable} \\
 & | & \lambda x\colon \phi.M & \text{Abstraction} \\
 & | & M\,M & \text{Application} \\
 & | & !^w M & \text{w} - \text{Promotion} \\
 & | & !^c M & \text{c} - \text{Promotion} \\
 & | & !M & ! - \text{Promotion} \\
 & | & \mathsf{let}\ M\ \mathsf{be}\ !^w x\ \mathsf{in}\ M & \text{w} - \text{Dereliction} \\
 & | & \mathsf{let}\ M\ \mathsf{be}\ !^c x\ \mathsf{in}\ M & \text{c} - \text{Dereliction} \\
 & | & \mathsf{let}\ M\ \mathsf{be}\ !x\ \mathsf{in}\ M & ! - \text{Dereliction}
\end{array}
$$

[6] A similar calculus has been (independently) considered by Maraist [21]. Further details of this calculus are given in §7.

where x is taken from some countable set of variables. Typing judgements are written $\Gamma;\Delta;\Sigma;\Theta \rhd M\colon\phi$ where Γ and Θ are multisets of pairs of variables and types, and Δ and Σ are sets of pairs of variables and types. The rules for forming valid typing judgements are given below.

$$\frac{}{\Gamma;-;\Sigma;x\colon\phi \rhd x\colon\phi}\text{Identity} \qquad \frac{}{\Gamma,x\colon\phi;-;\Sigma;- \rhd x\colon\phi}\text{Identity}^{w}$$

$$\frac{}{\Gamma;x\colon\phi;\Sigma;- \rhd x\colon\phi}\text{Identity}^{c} \qquad \frac{}{\Gamma;-;\Sigma,x\colon\phi;- \rhd x\colon\phi}\text{Identity}^{!}$$

$$\frac{\Gamma;\Delta;\Sigma;\Theta,x\colon\phi \rhd M\colon\psi}{\Gamma;\Delta;\Sigma;\Theta \rhd \lambda x\colon\phi.M\colon\phi\multimap\psi}\multimap_{\mathcal{I}}$$

$$\frac{\Gamma;\Delta,\Delta';\Sigma;\Theta \rhd M\colon\phi\multimap\psi \qquad \Gamma';\Delta,\Delta'';\Sigma;\Theta' \rhd N\colon\phi}{\Gamma,\Gamma';\Delta,\Delta',\Delta'';\Sigma;\Theta,\Theta' \rhd MN\colon\psi}\multimap_{\mathcal{E}}$$

$$\frac{\Gamma;-;\Sigma;- \rhd M\colon\phi}{\Gamma;-;\Sigma;-\rhd !^{w}M\colon !^{w}\phi}!^{w}_{\mathcal{I}} \quad \frac{\Gamma;\Delta,\Delta';\Sigma;\Theta \rhd M\colon !^{w}\phi \qquad \Gamma',x\colon\phi;\Delta,\Delta'';\Sigma;\Theta' \rhd N\colon\psi}{\Gamma,\Gamma';\Delta,\Delta',\Delta'';\Sigma;\Theta,\Theta' \rhd \mathsf{let}\ M\ \mathsf{be}\ !^{w}x\ \mathsf{in}\ N\colon\psi}!^{w}_{\mathcal{E}}$$

$$\frac{-;\Delta;\Sigma;- \rhd M\colon\phi}{\Gamma;\Delta;\Sigma;-\rhd !^{c}M\colon !^{c}\phi}!^{c}_{\mathcal{I}} \quad \frac{\Gamma;\Delta,\Delta';\Sigma;\Theta \rhd M\colon !^{c}\phi \qquad \Gamma';\Delta,x\colon\phi,\Delta'';\Sigma;\Theta' \rhd N\colon\psi}{\Gamma,\Gamma';\Delta,\Delta',\Delta'';\Sigma;\Theta,\Theta' \rhd \mathsf{let}\ M\ \mathsf{be}\ !^{c}x\ \mathsf{in}\ N\colon\psi}!^{c}_{\mathcal{E}}$$

$$\frac{-;-;\Sigma;- \rhd M\colon\phi}{\Gamma;-;\Sigma;-\rhd !M\colon !\phi}!_{\mathcal{I}} \quad \frac{\Gamma;\Delta,\Delta';\Sigma;\Theta \rhd M\colon !\phi \qquad \Gamma';\Delta,\Delta'';\Sigma,x\colon\phi;\Theta' \rhd N\colon\psi}{\Gamma,\Gamma';\Delta,\Delta',\Delta'';\Sigma;\Theta,\Theta' \rhd \mathsf{let}\ M\ \mathsf{be}\ !x\ \mathsf{in}\ N\colon\psi}!_{\mathcal{E}}$$

In §3.3 a number of admissible rules were identified. Here they are repeated but annotated with the corresponding linear λ-terms.

$$\frac{\Gamma;\Delta;\Sigma;\Theta \rhd M\colon\phi}{\Gamma,x\colon\psi;\Delta;\Sigma;\Theta \rhd M\colon\phi} \quad \frac{\Gamma;\Delta;\Sigma;\Theta \rhd M\colon\phi}{\Gamma;\Delta;\Sigma,x\colon\psi;\Theta \rhd M\colon\phi}$$

$$\frac{\Gamma;\Delta;\Sigma;\Theta \rhd M\colon\psi}{\Gamma;\Delta;\Sigma;\Theta,x\colon !^{w}\phi \rhd \mathsf{let}\ x\ \mathsf{be}\ !^{w}y\ \mathsf{in}\ M\colon\psi}\,(y\notin \textsc{fv}(M))$$

$$\frac{\Gamma;\Delta,x\colon\psi,y\colon\psi;\Sigma;\Theta \rhd M\colon\phi}{\Gamma;\Delta,z\colon\psi;\Sigma;\Theta \rhd M[x,y:=z]\colon\phi} \quad \frac{\Gamma;\Delta;\Sigma,x\colon\psi,y\colon\psi;\Theta \rhd M\colon\phi}{\Gamma;\Delta;\Sigma,z\colon\psi;\Theta \rhd M[x,y:=z]\colon\phi}$$

$$\frac{\Gamma;\Delta;\Sigma;\Theta,x\colon !^{c}\phi,y\colon !^{c}\phi \rhd M\colon\psi}{\Gamma;\Delta;\Sigma;\Theta,z\colon !^{c}\phi \rhd \mathsf{let}\ z\ \mathsf{be}\ !^{c}w\ \mathsf{in}\ M[x,y:=!^{c}w]\colon\psi}$$

$$\frac{\Gamma,x\colon\phi;\Delta;\Sigma;\Theta \rhd M\colon\psi}{\Gamma;\Delta;\Sigma,y\colon\phi;\Theta \rhd M[x:=y]\colon\psi} \qquad \frac{\Gamma;\Delta,x\colon\phi;\Sigma;\Theta \rhd M\colon\psi}{\Gamma;\Delta;\Sigma,y\colon\phi;\Theta \rhd M[x:=y]\colon\psi}$$

$$\frac{\Gamma;\Delta;\Sigma;\Theta,x\colon\phi \rhd M\colon\psi}{\Gamma,y\colon\phi;\Delta;\Sigma;\Theta \rhd M[x:=y]\colon\psi} \qquad \frac{\Gamma;\Delta;\Sigma;\Theta,x\colon\phi \rhd M\colon\psi}{\Gamma;\Delta,y\colon\phi;\Sigma;\Theta \rhd M[x:=y]\colon\psi}$$

$$\frac{\Gamma;\Delta;\Sigma;\Theta \rhd M\colon !\phi}{\Gamma;\Delta;\Sigma;\Theta \rhd \mathsf{let}\ M\ \mathsf{be}\ !^{w}x\ \mathsf{in}\ !^{w}x\colon !^{w}\phi} \quad \frac{\Gamma;\Delta;\Sigma;\Theta \rhd M\colon !\phi}{\Gamma;\Delta;\Sigma;\Theta \rhd \mathsf{let}\ M\ \mathsf{be}\ !^{c}x\ \mathsf{in}\ !^{c}x\colon !^{c}\phi}$$

Associated with the linear λ-terms are a number of β-reduction rules. These correspond to the elimination of unnecessary detours (created by introduction-elimination pairs) in the natural deduction formulation. The reduction rules are as follows.

$$\begin{array}{rcl}
(\lambda x.N)M & \rightsquigarrow_\beta & N[x := M] \\
\mathsf{let}\ !^w M\ \mathsf{be}\ !^w x\ \mathsf{in}\ N & \rightsquigarrow_\beta & N[x := M] \\
\mathsf{let}\ !^c M\ \mathsf{be}\ !^c x\ \mathsf{in}\ N & \rightsquigarrow_\beta & N[x := M] \\
\mathsf{let}\ !M\ \mathsf{be}\ !x\ \mathsf{in}\ N & \rightsquigarrow_\beta & N[x := M]
\end{array}$$

As expected, these rules satisfy the so-called subject reduction property.

Lemma 6. *If* $\Gamma;\Delta;\Sigma;\Theta \rhd M{:}\,\phi$ *and* $M \rightsquigarrow_\beta N$ *then* $\Gamma;\Delta;\Sigma;\Theta \rhd N{:}\,\phi$.

Moreover we can show that all sequences of β-reductions are finite.

Theorem 6. *The reduction system is strongly normalising.*

Proof. This can be proved by conventional means (cf. [10, Theorem 18]) or by a variant of Benton's translation into System F [6]. ■

There are a number of additional reduction rules, the commuting conversions, caused by the 'parasitic formula' [17, §10.1] in the modality elimination rules. As Girard has shown they can also be seen as naturally arising when considering the subformula property [17, §10.2]. For this calculus there are twelve commuting conversions, which are as follows.

$$\begin{array}{rcl}
(\mathsf{let}\ M\ \mathsf{be}\ !^w x\ \mathsf{in}\ N)P & \rightsquigarrow_c & \mathsf{let}\ M\ \mathsf{be}\ !^w x\ \mathsf{in}\ (NP) \\
(\mathsf{let}\ M\ \mathsf{be}\ !^c x\ \mathsf{in}\ N)P & \rightsquigarrow_c & \mathsf{let}\ M\ \mathsf{be}\ !^c x\ \mathsf{in}\ (NP) \\
(\mathsf{let}\ M\ \mathsf{be}\ !x\ \mathsf{in}\ N)P & \rightsquigarrow_c & \mathsf{let}\ M\ \mathsf{be}\ !x\ \mathsf{in}\ (NP) \\
\mathsf{let}\ (\mathsf{let}\ M\ \mathsf{be}\ !^w x\ \mathsf{in}\ N)\ \mathsf{be}\ !^w y\ \mathsf{in}\ P & \rightsquigarrow_c & \mathsf{let}\ M\ \mathsf{be}\ !^w x\ \mathsf{in}\ (\mathsf{let}\ N\ \mathsf{be}\ !^w y\ \mathsf{in}\ P) \\
\mathsf{let}\ (\mathsf{let}\ M\ \mathsf{be}\ !^w x\ \mathsf{in}\ N)\ \mathsf{be}\ !^c y\ \mathsf{in}\ P & \rightsquigarrow_c & \mathsf{let}\ M\ \mathsf{be}\ !^w x\ \mathsf{in}\ (\mathsf{let}\ N\ \mathsf{be}\ !^c y\ \mathsf{in}\ P) \\
\mathsf{let}\ (\mathsf{let}\ M\ \mathsf{be}\ !^w x\ \mathsf{in}\ N)\ \mathsf{be}\ !y\ \mathsf{in}\ P & \rightsquigarrow_c & \mathsf{let}\ M\ \mathsf{be}\ !^w x\ \mathsf{in}\ (\mathsf{let}\ N\ \mathsf{be}\ !y\ \mathsf{in}\ P) \\
\mathsf{let}\ (\mathsf{let}\ M\ \mathsf{be}\ !^c x\ \mathsf{in}\ N)\ \mathsf{be}\ !^w y\ \mathsf{in}\ P & \rightsquigarrow_c & \mathsf{let}\ M\ \mathsf{be}\ !^c x\ \mathsf{in}\ (\mathsf{let}\ N\ \mathsf{be}\ !^w y\ \mathsf{in}\ P) \\
\mathsf{let}\ (\mathsf{let}\ M\ \mathsf{be}\ !^c x\ \mathsf{in}\ N)\ \mathsf{be}\ !^c y\ \mathsf{in}\ P & \rightsquigarrow_c & \mathsf{let}\ M\ \mathsf{be}\ !^c x\ \mathsf{in}\ (\mathsf{let}\ N\ \mathsf{be}\ !^c y\ \mathsf{in}\ P) \\
\mathsf{let}\ (\mathsf{let}\ M\ \mathsf{be}\ !^c x\ \mathsf{in}\ N)\ \mathsf{be}\ !y\ \mathsf{in}\ P & \rightsquigarrow_c & \mathsf{let}\ M\ \mathsf{be}\ !^c x\ \mathsf{in}\ (\mathsf{let}\ N\ \mathsf{be}\ !y\ \mathsf{in}\ P) \\
\mathsf{let}\ (\mathsf{let}\ M\ \mathsf{be}\ !x\ \mathsf{in}\ N)\ \mathsf{be}\ !^w y\ \mathsf{in}\ P & \rightsquigarrow_c & \mathsf{let}\ M\ \mathsf{be}\ !x\ \mathsf{in}\ (\mathsf{let}\ N\ \mathsf{be}\ !^w y\ \mathsf{in}\ P) \\
\mathsf{let}\ (\mathsf{let}\ M\ \mathsf{be}\ !x\ \mathsf{in}\ N)\ \mathsf{be}\ !^c y\ \mathsf{in}\ P & \rightsquigarrow_c & \mathsf{let}\ M\ \mathsf{be}\ !x\ \mathsf{in}\ (\mathsf{let}\ N\ \mathsf{be}\ !^c y\ \mathsf{in}\ P) \\
\mathsf{let}\ (\mathsf{let}\ M\ \mathsf{be}\ !x\ \mathsf{in}\ N)\ \mathsf{be}\ !y\ \mathsf{in}\ P & \rightsquigarrow_c & \mathsf{let}\ M\ \mathsf{be}\ !x\ \mathsf{in}\ (\mathsf{let}\ N\ \mathsf{be}\ !y\ \mathsf{in}\ P)
\end{array}$$

Theorem 7. *The reduction system (extended with the commuting conversions) is confluent.*

Proof. This can be proved by standard techniques [4]. ■

5.2 Typed Combinators

The Curry-Howard correspondence also relates axiomatic formulations with systems of typed combinators. The prototypical example is the correspondence between the axiomatic formulation of propositional minimal logic and the `S,K,I`-combinatory system. Again this relationship has also been studied quite thoroughly for **ILL** [10]. In this section I shall describe the combinatory system which corresponds to the axiomatic formulation of §4. In the next section I consider a multi-context formulation of this combinatory system.

Raw combinatory terms are given by the following grammar.

$$
\begin{array}{rcll}
S & ::= & x & \text{Variable} \\
& | & \mathtt{I}, \mathtt{B}, \mathtt{C}, \mathtt{W}, \mathtt{k}^w, \mathtt{k}^!, \mathtt{order}^w, \mathtt{order}^c, & \\
& & \mathtt{T}^w, \mathtt{K}^w, \mathtt{4}^w, \mathtt{T}^c, \mathtt{K}^c, \mathtt{4}^c, \mathtt{T}^!, \mathtt{K}^!, \mathtt{4}^! & \text{Combinator} \\
& | & S\,S & \text{Application} \\
& | & !^w S & \text{w} - \text{Promotion} \\
& | & !^c S & \text{c} - \text{Promotion} \\
& | & !S & ! - \text{Promotion}
\end{array}
$$

where x is taken from some countable set of variables. Typing judgements are written $\Gamma \Rightarrow S\colon \phi$ where Γ is a multiset of pairs of variables and types. The rules for forming valid typing judgements are as follows.

$$
\frac{}{x\colon \phi \Rightarrow x\colon \phi}\ \text{Identity} \qquad \frac{}{\Rightarrow \mathtt{c}\colon \phi}\ \text{Combinator}
$$

$$
\frac{\Gamma \Rightarrow S\colon \phi \multimap \psi \qquad \Delta \Rightarrow T\colon \phi}{\Gamma, \Delta \Rightarrow ST\colon \psi}\ \text{Modus Ponens}
$$

$$
\frac{\Rightarrow S\colon \phi}{\Rightarrow !^w S\colon !^w \phi}\ !^w \qquad \frac{\Rightarrow S\colon \phi}{\Rightarrow !^c S\colon !^c \phi}\ !^c \qquad \frac{\Rightarrow S\colon \phi}{\Rightarrow !S\colon !\phi}\ !
$$

It is important to note the restriction on the last three rules: the combinatory term, S, must be completely closed (i.e. contain no free variables) for the rule to be validly applied.

Associated with these combinators is a notion of reduction (often called 'weak' reduction), which is written $\rightsquigarrow_w$.

$$
\begin{array}{rcl}
\mathtt{I}\,S & \rightsquigarrow_w & S \\
\mathtt{B}\,S\,T\,U & \rightsquigarrow_w & S\,(T\,U) \\
\mathtt{C}\,S\,T\,U & \rightsquigarrow_w & S\,U\,T \\
\mathtt{W}^c\,S\,T & \rightsquigarrow_w & S\,T\,T \\
\mathtt{W}^!\,S\,T & \rightsquigarrow_w & S\,T\,T \\
\mathtt{k}^w\,S\,T & \rightsquigarrow_w & S \\
\mathtt{k}^!\,S\,T & \rightsquigarrow_w & S \\
\mathtt{T}^w\,!^w S & \rightsquigarrow_w & S \\
\mathtt{4}^w\,!^w S & \rightsquigarrow_w & !^w !^w S \\
\mathtt{K}^w\,!^w S\,!^w T & \rightsquigarrow_w & !^w (S\,T) \\
\mathtt{T}^c\,!^c S & \rightsquigarrow_w & S \\
\mathtt{4}^c\,!^c S & \rightsquigarrow_w & !^c !^c S \\
\mathtt{K}^c\,!^c S\,!^c T & \rightsquigarrow_w & !^c (S\,T) \\
\mathtt{T}^!\,!S & \rightsquigarrow_w & S \\
\mathtt{4}^!\,!S & \rightsquigarrow_w & !!S \\
\mathtt{K}^!\,!S\,!T & \rightsquigarrow_w & !(S\,T) \\
\mathtt{order}^w !S & \rightsquigarrow_w & !^w S \\
\mathtt{order}^c !S & \rightsquigarrow_w & !^c S
\end{array}
$$

As expected these rules satisfy the subject reduction property.

Lemma 7. *If* $\Gamma \Rightarrow S\colon \phi$ *and* $S \rightsquigarrow_w T$ *then* $\Gamma \Rightarrow T\colon \phi$.

A constructive proof of the deduction theorem (Theorem 4) corresponds, via the Curry-Howard correspondence, to an abstraction algorithm for removing variables from a combinatory term. Abstracting a variable x from a term S (where x occurs free in S) is written $[x]S$[7] and can be defined as follows (where $\mathrm{FV}(S)$ denotes the set of free variables of a term S).

$$\begin{array}{rcl} [x]x & \stackrel{\mathrm{def}}{=} & \mathtt{I} \\ {[x]ST} & \stackrel{\mathrm{def}}{=} & \begin{cases} \mathtt{C}\,([x]S)\,T & x \in \mathrm{FV}(S) \\ \mathtt{B}\,S\,([x]T) & x \in \mathrm{FV}(T) \end{cases} \end{array}$$

Lemma 8. *If $\Gamma, x{:}\,\phi \Rightarrow S{:}\,\psi$ then $\Gamma \Rightarrow [x]S{:}\,\phi \multimap \psi$.*

Theorem 5 shows that there is an equivalence between the natural deduction formulation and the axiomatic formulation. Using the Curry-Howard correspondence, this can be lifted to a translation of the linear λ-terms from §5.1 to the typed combinators of this section. This translation, $\llbracket - \rrbracket$, is defined as follows.

$$\begin{array}{rcl}
\llbracket \vec{w}{:}\,\Gamma; -; \vec{e}{:}\,\Sigma; x{:}\,\phi \rhd x{:}\,\phi \rrbracket & \stackrel{\mathrm{def}}{=} & \mathbf{disc}^{w}(\vec{w}, \mathbf{disc}^{!}(\vec{e}, x)) \\
\llbracket \vec{w}{:}\,\Gamma, x{:}\,\phi; -; \vec{e}{:}\,\Sigma; - \rhd x{:}\,\phi \rrbracket & \stackrel{\mathrm{def}}{=} & \mathbf{disc}^{w}(\vec{w}, \mathbf{disc}^{!}(\vec{e}, \mathtt{T}^{w}\,x)) \\
\llbracket \vec{w}{:}\,\Gamma; x{:}\,\phi; \vec{e}{:}\,\Sigma; - \rhd x{:}\,\phi \rrbracket & \stackrel{\mathrm{def}}{=} & \mathbf{disc}^{w}(\vec{w}, \mathbf{disc}^{!}(\vec{e}, \mathtt{T}^{c}\,x)) \\
\llbracket \vec{w}{:}\,\Gamma; -; : e{:}\,\Sigma, x{:}\,\phi; - \rhd x{:}\,\phi \rrbracket & \stackrel{\mathrm{def}}{=} & \mathbf{disc}^{w}(\vec{w}, \mathbf{disc}^{!}(\vec{e}, \mathtt{T}^{!}\,x)) \\
\llbracket \Gamma; \Delta; \Sigma; \Theta \rhd \lambda x{:}\,\phi.M{:}\,\phi \multimap \psi \rrbracket & \stackrel{\mathrm{def}}{=} & [x]\llbracket M \rrbracket \\
\llbracket \Gamma; \Delta; \Sigma; \Theta \rhd MN{:}\,\psi \rrbracket & \stackrel{\mathrm{def}}{=} & \llbracket M \rrbracket \llbracket N \rrbracket \\
\llbracket \Gamma; \Delta; \Sigma; \Theta \rhd \mathsf{let}\ M\ \mathsf{be}\ !^{w}x\ \mathsf{in}\ N{:}\,\psi \rrbracket & \stackrel{\mathrm{def}}{=} & (\mathtt{B}\,([x]\llbracket N \rrbracket)\,\mathtt{T}^{w})\,\llbracket M \rrbracket \\
\llbracket \vec{x}{:}\,\Gamma; -; \vec{y}{:}\,\Sigma; - \rhd !^{w}M{:}\,!^{w}\phi \rrbracket & \stackrel{\mathrm{def}}{=} & (\mathtt{B}\,(\mathtt{K}^{w}\,(\cdots(\mathtt{B}\,\mathtt{K}^{w}\ S)\,(\mathtt{B}\,\mathtt{order}^{w}\ \mathtt{4}^{!}))y_1 \cdots))\,(\mathtt{B}\,\mathtt{order}^{w}\ \mathtt{4}^{!})\,y_n \\
& & \text{where} \\
& & \quad S \stackrel{\mathrm{def}}{=} (\mathtt{B}\,(\mathtt{K}^{w}\,(\cdots(\mathtt{B}\,(\mathtt{K}^{w}\,T)\,\mathtt{4}^{w})x_1\cdots))\mathtt{4}^{w})x_n \\
& & \quad T \stackrel{\mathrm{def}}{=} !^{w}(\mathtt{B}\,([x_1]\cdots\mathtt{B}\,[x_n](\mathtt{B}\,([y_1]\cdots\mathtt{B}([y_n]\llbracket M \rrbracket)\mathtt{T}^{!}\cdots)\,\mathtt{T}^{!})\,\mathtt{T}^{w}\cdots)\mathtt{T}^{w}) \\
\llbracket \Gamma; \Delta; \Sigma; \Theta \rhd \mathsf{let}\ M\ \mathsf{be}\ !^{c}x\ \mathsf{in}\ N{:}\,\psi \rrbracket & \stackrel{\mathrm{def}}{=} & (\mathtt{B}\,([x]\llbracket N \rrbracket)\,\mathtt{T}^{c})\,\llbracket M \rrbracket \\
\llbracket \vec{w}{:}\,\Gamma; \vec{x}{:}\,\Delta; \vec{y}{:}\,\Sigma; - \rhd !^{c}M{:}\,!^{c}\phi \rrbracket & \stackrel{\mathrm{def}}{=} & \mathbf{disc}^{w}(\vec{w}, (\mathtt{B}\,(\mathtt{K}^{c}\,(\cdots(\mathtt{B}\,\mathtt{K}^{c}\ S)\,(\mathtt{B}\,\mathtt{order}^{c}\ \mathtt{4}^{!}))y_1 \cdots))\,(\mathtt{B}\,\mathtt{order}^{c}\ \mathtt{4}^{!})\,y_n) \\
& & \text{where} \\
& & \quad S \stackrel{\mathrm{def}}{=} (\mathtt{B}\,(\mathtt{K}^{c}\,(\cdots(\mathtt{B}\,(\mathtt{K}^{c}\,T)\,\mathtt{4}^{c})x_1\cdots))\mathtt{4}^{c})x_n \\
& & \quad T \stackrel{\mathrm{def}}{=} !^{c}(\mathtt{B}\,([x_1]\cdots\mathtt{B}\,[x_n](\mathtt{B}\,([y_1]\cdots\mathtt{B}([y_n]\llbracket M \rrbracket)\mathtt{T}^{!}\cdots)\,\mathtt{T}^{!})\,\mathtt{T}^{c}\cdots)\mathtt{T}^{c}) \\
\llbracket \Gamma; \Delta; \Sigma; \Theta \rhd \mathsf{let}\ M\ \mathsf{be}\ !x\ \mathsf{in}\ N{:}\,\psi \rrbracket & \stackrel{\mathrm{def}}{=} & (\mathtt{B}\,([x]\llbracket N \rrbracket)\,\mathtt{T}^{!})\,\llbracket M \rrbracket \\
\llbracket \vec{w}{:}\,\Gamma; -; \vec{y}{:}\,\Sigma; - \rhd !M{:}\,!\phi \rrbracket & \stackrel{\mathrm{def}}{=} & \mathbf{disc}^{w}(\vec{w}, (\mathtt{B}\,(\mathtt{K}^{!}\,(\cdots(\mathtt{B}\,(\mathtt{K}^{!}\ S)\,\mathtt{4}^{!})y_1 \cdots))\,\mathtt{4}^{!})y_n) \\
& & \text{where} \\
& & \quad S \stackrel{\mathrm{def}}{=} !(\mathtt{B}\,([y_1](\cdots\mathtt{B}\,([y_n]\llbracket M \rrbracket)\,\mathtt{T}^{!}\cdots))\,\mathtt{T}^{!})
\end{array}$$

where I have made use of the following macros.

$$\begin{array}{rcl}
\mathbf{disc}^{w}(x_1, \ldots, x_n, S) & \stackrel{\mathrm{def}}{=} & \mathtt{k}^{w}\,(\cdots\mathtt{k}^{w}\,(\mathtt{k}^{w}\,S\,x_1)\,x_2\cdots)\,x_n \\
\mathbf{disc}^{!}(x_1, \ldots, x_n, S) & \stackrel{\mathrm{def}}{=} & \mathtt{k}^{!}\,(\cdots\mathtt{k}^{!}\,(\mathtt{k}^{!}\,S\,x_1)\,x_2\cdots)\,x_n
\end{array}$$

5.3 Multi-context Typed Combinators

It is quite possible to employ the multi-context techniques from §3.3 to the combinatory system from the previous section. The grammar for raw combinatory terms remains unchanged, but typing judgements are extended with multiple contexts. Typing judgements are now of the form $\Gamma; \Delta; \Sigma; \Theta \Rightarrow S{:}\,\phi$ where Γ and Θ are *multisets* of pairs of variables and types and Δ and Σ are *sets* of pairs of variables and types. As in §3.3, the judgement

$$\Gamma; \Delta; \Sigma; \Theta \Rightarrow S{:}\,\phi$$

[7]This is written $\lambda^{*}x.S$ by Barendregt [4].

is meant to represent to correspond to the judgement

$$!^w\Gamma, !^c\Delta, !\Sigma, \Theta \Rightarrow S{:}\,\phi$$

from the system in the previous section. Thus Γ is the $!^w$ context, Δ the $!^c$ context, Σ the ! context and Θ the linear context. The advantage of this multi-context formulation is that the rules for forming Promotion terms become less restrictive: before the term had to be completely closed before application. The rules for forming valid typing judgements are as follows.

$$\frac{}{\Gamma, x{:}\,\phi; -; \Sigma; - \Rightarrow x{:}\,\phi}\ \text{Identity}^w \qquad \frac{}{\Gamma; x{:}\,\phi; \Sigma; - \Rightarrow x{:}\,\phi}\ \text{Identity}^c$$

$$\frac{}{\Gamma; -; \Sigma, x{:}\,\phi; - \Rightarrow x{:}\,\phi}\ \text{Identity}^! \qquad \frac{}{\Gamma; -; \Sigma; x{:}\,\phi \Rightarrow x{:}\,\phi}\ \text{Identity}$$

$$\frac{}{\Gamma; -; \Gamma; - \Rightarrow \mathtt{c}{:}\,\phi}\ \text{Combinator}$$

$$\frac{\Gamma; \Delta, \Delta'; \Sigma; \Theta \Rightarrow S{:}\,\phi \multimap \psi \qquad \Gamma'; \Delta, \Delta''; \Sigma; \Theta' \Rightarrow T{:}\,\phi}{\Gamma, \Gamma'; \Delta, \Delta', \Delta''; \Sigma; \Theta, \Theta' \Rightarrow S\,T{:}\,\psi}\ \text{Modus Ponens}$$

$$\frac{\Gamma; -; \Sigma; - \Rightarrow S{:}\,\phi}{\Gamma; -; \Sigma; - \Rightarrow !^w S{:}\,!^w\phi}\ !^w \qquad \frac{-; \Delta; \Sigma; - \Rightarrow S{:}\,\phi}{\Gamma; \Delta; \Sigma; - \Rightarrow !^c S{:}\,!^c\phi}\ !^c$$

$$\frac{-; -; \Sigma; - \Rightarrow S{:}\,\phi}{\Gamma; -; \Sigma; - \Rightarrow !S{:}\,!\phi}\ !$$

As a variable can now reside in one of four contexts there are now four different variable abstraction algorithms. Abstraction of the variable x from the combinatory term S is written as either $[x^w]S$, $[x^c]S$, $[x^!]S$ or $[x^l]S$ depending on whether x is taken from the $!^w$, $!^c$, ! or linear context, respectively. The algorithms are as follows.

$$\begin{array}{rcl}
[x^l]x & \stackrel{\text{def}}{=} & \mathtt{I} \\
[x^l]S\,T & \stackrel{\text{def}}{=} & \begin{cases} \mathtt{C}\,([x^l]S)\,T & x \in \text{FV}(S) \\ \mathtt{B}\,S\,([x^l]T) & x \in \text{FV}(T) \end{cases} \\
\\
[x^w]y & \stackrel{\text{def}}{=} & \begin{cases} \mathtt{T}^w & x = y \\ \mathtt{k}^w y & x \neq y \end{cases} \\
[x^w]S\,T & \stackrel{\text{def}}{=} & \begin{cases} \mathtt{C}\,([x^l]S)\,T & x \in \text{FV}(S) \\ \mathtt{B}\,S\,([x^l]T) & x \in \text{FV}(T) \end{cases} \\
[x^w](!^w S) & \stackrel{\text{def}}{=} & \mathtt{B}\,(\mathtt{K}^w\ !^w[x^w]S)\ \mathtt{4}^w \\
\\
[x^c]x & \stackrel{\text{def}}{=} & \mathtt{T}^c \\
[x^c]S\,T & \stackrel{\text{def}}{=} & \mathtt{W}^c\,(\mathtt{C}\,(\mathtt{B}\,\mathtt{B}\,[x^c]S)\,[x^c]T) \\
[x^c](!^c S) & \stackrel{\text{def}}{=} & \mathtt{B}\,(\mathtt{K}^c\ !^c[x^c]S)\ \mathtt{4}^c \\
\\
[x^!]x & \stackrel{\text{def}}{=} & \begin{cases} \mathtt{T}^! & x = y \\ \mathtt{k}^! y & x \neq y \end{cases} \\
[x^!]S\,T & \stackrel{\text{def}}{=} & \mathtt{W}^!\,(\mathtt{C}\,(\mathtt{B}\,\mathtt{B}\,[x^!]S)\,[x^!]T) \\
[x^!](!S) & \stackrel{\text{def}}{=} & \mathtt{B}\,(\mathtt{K}^!\ ![x^!]S)\ \mathtt{4}^! \\
[x^!](!^w S) & \stackrel{\text{def}}{=} & \mathtt{B}\,(\mathtt{K}^w\ !^w([x^!]S))\,(\mathtt{B}\,\mathtt{order}^w\ \mathtt{4}^!) \\
[x^!](!^c S) & \stackrel{\text{def}}{=} & \mathtt{B}\,(\mathtt{K}^c\ !^c([x^!]S))\,(\mathtt{B}\,\mathtt{order}^c\ \mathtt{4}^!)
\end{array}$$

These abstraction algorithms preserve the typing judgements in the expected way.

Lemma 9.

1. *If* $\Gamma, x{:}\,\phi; \Delta; \Sigma; \Theta \Rightarrow S{:}\,\psi$ *then* $\Gamma; \Delta; \Sigma; \Theta \Rightarrow [x^w]S{:}\,\phi \multimap \psi$.
2. *If* $\Gamma; \Delta, x{:}\,\phi; \Sigma; \Theta \Rightarrow S{:}\,\psi$ *then* $\Gamma; \Delta; \Sigma; \Theta \Rightarrow [x^c]S{:}\,\phi \multimap \psi$.
3. *If* $\Gamma; \Delta; \Sigma, x{:}\,\phi; \Theta \Rightarrow S{:}\,\psi$ *then* $\Gamma; \Delta; \Sigma; \Theta \Rightarrow [x^!]S{:}\,\phi \multimap \psi$.
4. *If* $\Gamma; \Delta; \Sigma; \Theta, x{:}\,\phi \Rightarrow S{:}\,\psi$ *then* $\Gamma; \Delta; \Sigma; \Theta \Rightarrow [x^l]S{:}\,\phi \multimap \psi$.

One advantage of these multi-context combinators is that the translation of the linear λ-terms of §5.1 becomes quite succinct.

$$
\begin{array}{rcl}
[\![\Gamma; -; \Sigma; x{:}\,\phi \rhd x{:}\,\phi]\!] & \stackrel{\text{def}}{=} & x \\
[\![\Gamma, x{:}\,\phi; -; \Sigma; - \rhd x{:}\,\phi]\!] & \stackrel{\text{def}}{=} & x \\
[\![\Gamma; x{:}\,\phi; \Sigma; - \rhd x{:}\,\phi]\!] & \stackrel{\text{def}}{=} & x \\
[\![\Gamma; -; \Sigma, x{:}\,\phi; - \rhd x{:}\,\phi]\!] & \stackrel{\text{def}}{=} & x \\
[\![\Gamma; \Delta; \Sigma; \Theta \rhd \lambda x{:}\,\phi.M{:}\,\phi \multimap \psi]\!] & \stackrel{\text{def}}{=} & [x^l][\![M]\!] \\
[\![\Gamma; \Delta; \Sigma; \Theta \rhd MN{:}\,\psi]\!] & \stackrel{\text{def}}{=} & [\![M]\!][\![N]\!] \\
[\![\Gamma; \Delta; \Sigma; \Theta \rhd \mathsf{let}\ M\ \mathsf{be}\ !^w x\ \mathsf{in}\ N{:}\,\psi]\!] & \stackrel{\text{def}}{=} & ([x^w][\![N]\!])\ [\![M]\!] \\
[\![\vec{x}{:}\,\Gamma; -; \vec{y}{:}\,\Sigma; - \rhd !^w M{:}\,!^w\phi]\!] & \stackrel{\text{def}}{=} & !^w[\![M]\!] \\
[\![\Gamma; \Delta; \Sigma; \Theta \rhd \mathsf{let}\ M\ \mathsf{be}\ !^c x\ \mathsf{in}\ N{:}\,\psi]\!] & \stackrel{\text{def}}{=} & ([x^c][\![N]\!])\ [\![M]\!] \\
[\![\Gamma; \vec{x}{:}\,\Delta; \vec{y}{:}\,\Sigma; - \rhd !^c M{:}\,!^c\phi]\!] & \stackrel{\text{def}}{=} & !^c[\![M]\!] \\
[\![\Gamma; \Delta; \Sigma; \Theta \rhd \mathsf{let}\ M\ \mathsf{be}\ !x\ \mathsf{in}\ N{:}\,\psi]\!] & \stackrel{\text{def}}{=} & ([x^!][\![N]\!])\ [\![M]\!] \\
[\![\Gamma; -; \vec{y}{:}\,\Sigma; - \rhd !M{:}\,!\phi]\!] & \stackrel{\text{def}}{=} & ![\![M]\!]
\end{array}
$$

6 Classical Systems

This paper has only considered the intuitionistic fragment of linear logic. The methods described herein can be transferred quite simply to the classical fragment. The duality inherent in the classical setting means that for each modality there is its dual, which in the linear setting is written ?. Thus considering the classical version of the three modality logic of §2.1, the grammar for formulae is now

$$
\begin{array}{rcl l l}
\phi & ::= & p & \mid \phi \parr \phi & \mid \phi^\perp \\
& \mid & !^w\phi & \mid !^c\phi & \mid !\phi \\
& \mid & ?^w\phi & \mid ?^c\phi & \mid ?\phi
\end{array}
$$

(Instead of linear implication, $\multimap$, I shall consider the multiplicative disjunction, $\parr$, and negation. The implication can be recovered in the usual way, i.e. $\phi \multimap \psi \stackrel{\text{def}}{=} \phi^\perp \parr \psi$.) The modalities are related via the negation, i.e.

$$
\begin{array}{rcl}
(!^w\phi)^\perp & \equiv & ?^w\phi^\perp \\
(!^c\phi)^\perp & \equiv & ?^c\phi^\perp \\
(!\phi)^\perp & \equiv & ?\phi^\perp
\end{array}
$$

A sequent is now of the form $\Gamma \vdash \Delta$, where both Γ and Δ denote multisets of formulae. The sequent calculus formulation is then as follows.

$$p \vdash p$$

$$\frac{\Gamma \vdash \phi, \Delta \qquad \Gamma', \phi \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\ \text{Cut}$$

$$\frac{\Gamma, \phi \vdash \Delta \quad \Gamma', \psi \vdash \Delta'}{\Gamma, \phi \parr \psi, \Gamma' \vdash \Delta, \Delta'}\ \parr_{\mathcal{L}} \qquad \frac{\Gamma \vdash \phi, \psi}{\Gamma \vdash \phi \parr \psi}\ \parr_{\mathcal{R}}$$

$$\frac{\Gamma \vdash \phi, \Delta}{\Gamma, \phi^{\perp} \vdash \Delta}\ \perp_{\mathcal{L}} \qquad \frac{\Gamma, \phi \vdash \Delta}{\Gamma \vdash \Delta, \phi^{\perp}}\ \perp_{\mathcal{R}}$$

$$\frac{\Gamma, \phi \vdash \Delta}{\Gamma, !^{w}\phi \vdash \Delta}\ !^{w}_{\mathcal{L}} \qquad \frac{\Gamma \vdash \Delta, \phi}{\Gamma \vdash \Delta, ?^{w}\phi}\ ?^{w}_{\mathcal{R}}$$

$$\frac{!^{w}\Gamma, !\Gamma' \vdash \psi, ?^{w}\Delta, ?\Delta'}{!^{w}\Gamma, !\Gamma' \vdash !^{w}\psi, ?^{w}\Delta, ?\Delta'}\ !^{w}_{\mathcal{R}} \qquad \frac{!^{w}\Gamma, !\Gamma', \phi \vdash ?^{w}\Delta, ?\Delta'}{!^{w}\Gamma, !\Gamma', ?^{w}\phi \vdash ?^{w}\Delta, ?\Delta'}\ ?^{w}_{\mathcal{L}}$$

$$\frac{\Gamma, \phi \vdash \Delta}{\Gamma, !^{c}\phi \vdash \Delta}\ !^{c}_{\mathcal{L}} \qquad \frac{\Gamma \vdash \Delta, \phi}{\Gamma \vdash \Delta, ?^{c}\phi}\ ?^{c}_{\mathcal{R}}$$

$$\frac{!^{c}\Gamma, !\Gamma' \vdash \psi, ?^{c}\Delta, ?\Delta'}{!^{c}\Gamma, !\Gamma' \vdash !^{c}\psi, ?^{c}\Delta, ?\Delta'}\ !^{c}_{\mathcal{R}} \qquad \frac{!^{c}\Gamma, !\Gamma', \phi \vdash ?^{c}\Delta, ?\Delta'}{!^{c}\Gamma, !\Gamma', ?^{c}\phi \vdash ?^{c}\Delta, ?\Delta'}\ ?^{c}_{\mathcal{L}}$$

$$\frac{\Gamma, \phi \vdash \psi}{\Gamma, !\phi \vdash \psi}\ !_{\mathcal{L}} \qquad \frac{\Gamma \vdash \Delta, \phi}{\Gamma \vdash \Delta, ?\phi}\ ?_{\mathcal{R}}$$

$$\frac{!\Delta \vdash \psi, ?\Delta}{!\Delta \vdash !\psi, ?\Delta}\ !_{\mathcal{R}} \qquad \frac{!\Gamma, \phi \vdash ?\Delta}{!\Gamma, ?\phi \vdash ?\Delta}\ ?_{\mathcal{L}}$$

$$\frac{\Gamma \vdash \Delta}{\Gamma, !^{w}\phi \vdash \Delta}\ \text{Weakening}_{\mathcal{L}} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, ?^{w}\phi}\ \text{Weakening}_{\mathcal{R}}$$

$$\frac{\Gamma \vdash \Delta}{\Gamma, !\phi \vdash \Delta}\ \text{Weakening}^{!}_{\mathcal{L}} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, ?\phi}\ \text{Weakening}^{!}_{\mathcal{R}}$$

$$\frac{\Gamma, !^{c}\phi, !^{c}\phi \vdash \Delta}{\Gamma, !^{c}\phi \vdash \Delta}\ \text{Contraction}_{\mathcal{L}} \qquad \frac{\Gamma \vdash \Delta, ?^{c}\phi, ?^{c}\phi}{\Gamma \vdash \Delta, ?^{c}\phi}\ \text{Contraction}_{\mathcal{L}}$$

$$\frac{\Gamma, !\phi, !\phi \vdash \Delta}{\Gamma, !\phi \vdash \Delta}\ \text{Contraction}^{!}_{\mathcal{L}} \qquad \frac{\Gamma \vdash \Delta, ?\phi, ?\phi}{\Gamma \vdash \Delta, ?\phi}\ \text{Contraction}^{!}_{\mathcal{L}}$$

Again it can be shown that this formulation satisfies a cut-elimination property. Two-sided sequents $\Gamma \vdash \Delta$ can be rewritten as a one-sided sequent $\vdash \Gamma^{\perp}, \Delta$. This allows for a more compact presentation and leads directly to a proof net formulation. This is left to the interested reader.

7 Conclusions

This paper has considered logics (principally intuitionistic linear logic) with multiple modalities. An important example is where the structural rules of Weakening and Contraction are permitted via separate modalities. Jacobs [20] has shown that semantically this situation occurs quite naturally. I have shown how one can give both sequent calculus, natural deduction and axiomatic formulations of such logics.

The λ-calculus of §5.1 is a language which distinguishes explicitly at the type and term level between those objects which are used exactly once, those which may be garbaged and those which may be copied. It seems likely that this sort of language could have real practical use. Many modern functional compilers, e.g. TIL [25] and MLJ [8], use explicit type/term-annotations to assist in optimisations. One operational detail suggests itself immediately. In operational treatments of linear λ-calculi [1, 11], one is interested in an evaluation relation, $M \Downarrow v$, which relates closed terms (programs), M, and values, v. This evaluation relation is normally defined using some form of structured operational semantics [23]. In treatments of linear λ-calculi the ! is treated as a closure, i.e.

$$!M \Downarrow !M$$

(one never evaluates under a !). The intuition is that an object $!M$ could be used any number of times, including zero, and so it may not be wise to evaluate its body (M). In the multiple modality calculus there is a more refined picture. An object $!^w M$ may not be used, but an object $!^c M$ is definitely used. Thus this suggests the following three rules.

$$!^w M \Downarrow !^w M \qquad \frac{M \Downarrow v}{!^c M \Downarrow !^c v} \qquad !M \Downarrow !M$$

A detailed study of the operational theory and an investigation of the practical applications of this linear λ-calculus will appear in joint work with A.M. Pitts [13].

Maraist [21] has also (independently) considered a linear λ-calculus, related to that given in §5.1.[8] One difference is that he does not have a distinct ! modality, but rather considers both combinations $!^w!^c$ and $!^c!^w$ to be equivalent and equal to !. Thus typing judgements are still of the form $\Gamma; \Delta; \Sigma; \Theta \rhd M : \phi$, but there is no introduction rule for ! and two elimination rules, i.e.

$$\frac{\Gamma; \Delta, \Delta'; \Sigma; \Theta \rhd M : !^w!^c\phi \qquad \Gamma'; \Delta, \Delta''; x : \phi, \Sigma; \Theta' \rhd N : \psi}{\Gamma, \Gamma'; \Delta, \Delta', \Delta''; \Sigma; \Theta, \Theta' \rhd \mathsf{let}\ M\ \mathsf{be}\ !x\ \mathsf{in}\ N : \psi}\ !^w!^c_{\mathcal{E}}$$

$$\frac{\Gamma; \Delta, \Delta'; \Sigma; \Theta \rhd M : !^c!^w\phi \qquad \Gamma'; \Delta, \Delta''; x : \phi, \Sigma; \Theta' \rhd N : \psi}{\Gamma, \Gamma'; \Delta, \Delta', \Delta''; \Sigma; \Theta, \Theta' \rhd \mathsf{let}\ M\ \mathsf{be}\ !x\ \mathsf{in}\ N : \psi}\ !^c!^w_{\mathcal{E}}$$

Of course, as Jacobs has shown [20], it is not always the case that $!^c!^w \cong !^w!^c$—presumably Maraist's system could be changed to reflect a different semantic picture. Another difference (cf. footnote 4) is that Maraist makes all four contexts multisets and has explicit weakening and contraction rules. He also includes a number of "dereliction" rules to move variables from one context to another. In fact they are unnecessary as they are all admissible rules (cf. Lemma 5). However the real purpose of Maraist's work is to study different translations

[8] I am grateful to Eike Ritter for bringing Maraist's work to my attention.

from the typed λ-calculus to his multiple modality calculus. A more detailed comparison with Maraist's calculus is interesting further work.

This paper has considered only *linear* logics with multiple modalities. However the techniques described apply to any logic with multiple **S4**-like modalities. For example, consider intuitionistic logic with three necessity modalities, written $\Box^a, \Box^b, \Box^c$ which are ordered as follows.

$$
\begin{tikzcd}
 & \Box^c \arrow[dl, dash] \arrow[dr, dash] & \\
\Box^a & & \Box^b
\end{tikzcd}
$$

A single-context natural deduction deduction formulation of this logic is, by following the techniques described in this paper, as follows.

$$\Gamma, \phi \vdash \phi$$

$$\frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \supset \psi} \supset_{\mathcal{I}} \qquad \frac{\Gamma \vdash \phi \supset \psi \qquad \Gamma \vdash \phi}{\Gamma \vdash \psi} \supset_{\mathcal{E}}$$

$$\frac{\Gamma \vdash \Box^a \psi_1 \quad \cdots \quad \Gamma \vdash \Box^a \psi_k \;\; \Gamma \vdash \Box^c \varphi_1 \quad \cdots \quad \Gamma \vdash \Box^c \varphi_l \qquad \Box^a \psi_1, \ldots, \Box^a \psi_k, \Box^c \varphi_1, \ldots, \Box^c \varphi_l \vdash \phi}{\Gamma, \Delta \vdash \Box^a \phi} \Box^a{}_{\mathcal{I}}$$

$$\frac{\Gamma \vdash \Box^a \phi}{\Gamma \vdash \phi} \Box^a{}_{\mathcal{E}}$$

$$\frac{\Gamma \vdash \Box^b \psi_1 \quad \cdots \quad \Gamma \vdash \Box^b \psi_k \;\; \Gamma \vdash \Box^c \varphi_1 \quad \cdots \quad \Gamma \vdash \Box^c \varphi_l \qquad \Box^b \psi_1, \ldots, \Box^b \psi_k, \Box^c \varphi_1, \ldots, \Box^c \varphi_l \vdash \phi}{\Gamma, \Delta \vdash \Box^b \phi} \Box^b{}_{\mathcal{I}}$$

$$\frac{\Gamma \vdash \Box^b \phi}{\Gamma \vdash \phi} \Box^b{}_{\mathcal{E}}$$

$$\frac{\Gamma \vdash \Box^c \psi_1 \quad \cdots \quad \Gamma \vdash \Box^c \psi_k \qquad \Box^c \psi_1, \ldots, \Box^c \psi_k \vdash \phi}{\Gamma, \Delta \vdash \Box^c \phi} \Box^c{}_{\mathcal{I}}$$

$$\frac{\Gamma \vdash \Box^c \phi}{\Gamma \vdash \phi} \Box^c{}_{\mathcal{E}}$$

A detailed study of intuitionistic and classical modal logics with multiple modalities is interesting future work.

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