What is a Categorical Model of Intuitionistic Linear Logic?

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Abstract. This paper re-addresses the old problem of providing a categorical model for Intuitionistic Linear Logic (**ILL**). In particular we compare the now standard model proposed by Seely to the lesser known one proposed by Benton, Bierman, Hyland and de Paiva. Surprisingly we find that Seely's model is *unsound* in that it does not preserve equality of proofs. We shall propose how to adapt Seely's definition so as to correct this problem and consider how this compares with the model due to Benton *et al.*

1 Intuitionistic Linear Logic

For the first part we shall consider only the multiplicative, exponential fragment of Intuitionistic Linear Logic (**MELL**). Rather than give a detailed description of the logic and associated term calculus we assume that the reader is familiar with other work [15, 5]. The sequent calculus formulation is originally due to Girard [9] and is given below.

$$\frac{}{A \vdash A}\ \mathit{Identity}$$

$$\frac{\Gamma \vdash B \qquad B, \Delta \vdash C}{\Gamma, \Delta \vdash C}\ \mathit{Cut}$$

$$\frac{\Gamma \vdash A}{\Gamma, I \vdash A}\ (I_{\mathcal{L}}) \qquad\qquad \frac{}{\vdash I}\ (I_{\mathcal{R}})$$

$$\frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}\ (\otimes_{\mathcal{L}}) \qquad \frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}\ (\otimes_{\mathcal{R}})$$

$$\frac{\Gamma \vdash A \qquad \Delta, B \vdash C}{\Gamma, \Delta, A \multimap B \vdash C}\ (\multimap_{\mathcal{L}}) \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}\ (\multimap_{\mathcal{R}})$$

$$\frac{\Gamma \vdash B}{\Gamma, !A \vdash B}\ \mathit{Weakening} \qquad \frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}\ \mathit{Contraction}$$

$$\frac{\Gamma, A \vdash B}{\Gamma, !A \vdash B}\ \mathit{Dereliction} \qquad \frac{!\Gamma \vdash A}{!\Gamma \vdash !A}\ \mathit{Promotion}$$

Sequents are written as $\Gamma \vdash A$, where A, B represent formulae and Γ, Δ represent multisets of formulae. Where Γ represents the multiset $A_1, \ldots, A_n$, then $!\Gamma$ is taken to represent the multiset $!A_1, \ldots, !A_n$.

The natural deduction presentation proved harder to formalize and early proposals [1, 15] failed to have the vital property of *closure under substitution*. A natural deduction system which has this property was given by Benton *et al.* [4] and is given below.

$$\frac{\begin{array}{c}[A^x]\\ \vdots\\ B\end{array}}{A \multimap B}\,(\multimap_{\mathcal{I}})_x \qquad \frac{\begin{array}{c}\vdots\\ A \multimap B\end{array} \quad \begin{array}{c}\vdots\\ A\end{array}}{B}\,(\multimap_{\mathcal{E}})$$

$$\frac{}{I}\,(I_{\mathcal{I}}) \qquad \frac{\begin{array}{c}\vdots\\ A\end{array} \quad \begin{array}{c}\vdots\\ I\end{array}}{A}\,(I_{\mathcal{E}})$$

$$\frac{\begin{array}{c}\vdots\\ A\end{array} \quad \begin{array}{c}\vdots\\ B\end{array}}{A \otimes B}\,(\otimes_{\mathcal{I}}) \qquad \frac{\begin{array}{c}\vdots\\ A \otimes B\end{array} \quad \begin{array}{c}[A^x]\ [B^y]\\ \vdots\\ C\end{array}}{C}\,(\otimes_{\mathcal{E}})_{x,y}$$

$$\frac{\begin{array}{c}\vdots\\ !B\end{array} \quad \begin{array}{c}\vdots\\ C\end{array}}{C}\,\textit{Weakening} \qquad \frac{\begin{array}{c}\vdots\\ !B\end{array} \quad \begin{array}{c}[!B^x]\ [!B^y]\\ \vdots\\ C\end{array}}{C}\,\textit{Contraction}_{x,y}$$

$$\frac{\begin{array}{c}\vdots\\ !B\end{array}}{B}\,\textit{Dereliction} \qquad \frac{\begin{array}{c}\vdots\\ !A_1\end{array} \ \ldots \ \begin{array}{c}\vdots\\ !A_n\end{array} \quad \begin{array}{c}[\![!A_1^{x_1} \ \cdots \ !A_n^{x_n}]\!]\\ \vdots\\ B\end{array}}{!B}\,\textit{Promotion}_{x1,\ldots,xn}$$

The main difference between this and earlier presentations is in the *Promotion* rule where here substitutions are 'built-in'.

The Curry-Howard correspondence [10] provides a systematic process for attaching names, or *terms*, to proof trees from the natural deduction formulation of a given constructive logic (a clear description is given by Gallier [7]). We can apply it to get the following term assignment system for **MELL**, which rather than presenting in a tree-like fashion, we choose to present in a sequent style.

$$\frac{}{x{:}\,A \triangleright x{:}\,A}\ \mathit{Identity}$$

$$\frac{\Gamma, x{:}\,A \triangleright M{:}\,B}{\Gamma \triangleright \lambda x{:}\,A.M{:}\,A \multimap B}\ (\multimap_{\mathcal{I}}) \qquad \frac{\Gamma \triangleright M{:}\,A \multimap B \qquad \Delta \triangleright N{:}\,A}{\Gamma, \Delta \triangleright MN{:}\,B}\ (\multimap_{\mathcal{E}})$$

$$\frac{}{\triangleright *{:}\,I}\ (I_{\mathcal{I}}) \qquad \frac{\Gamma \triangleright M{:}\,A \qquad \Delta \triangleright N{:}\,I}{\Gamma, \Delta \triangleright \mathsf{let}\ N\ \mathsf{be}\ *\ \mathsf{in}\ N{:}\,A}\ (I_{\mathcal{E}})$$

$$\frac{\Gamma \triangleright M{:}\,A \qquad \Delta \triangleright N{:}\,B}{\Gamma, \Delta \triangleright M \otimes N{:}\,A \otimes B}\ (\otimes_{\mathcal{I}}) \quad \frac{\Delta \triangleright M{:}\,A \otimes B \qquad \Gamma, x{:}\,A, y{:}\,B \triangleright N{:}\,C}{\Gamma, \Delta \triangleright \mathsf{let}\ M\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ N{:}\,C}\ (\otimes_{\mathcal{E}})$$

$$\frac{\Gamma_1 \triangleright M_1{:}\,!A_1 \quad \cdots \quad \Gamma_n \triangleright M_n{:}\,!A_n \qquad x_1{:}\,!A_1, \ldots, x_n{:}\,!A_n \triangleright N{:}\,B}{\Gamma_1, \ldots, \Gamma_n \triangleright \mathsf{promote}\ M_1, \ldots, M_n\ \mathsf{for}\ x_1, \ldots, x_n\ \mathsf{in}\ N{:}\,!B}\ \mathit{Promotion}$$

$$\frac{\Gamma \triangleright M{:}\,!A \qquad \Delta \triangleright N{:}\,B}{\Gamma, \Delta \triangleright \mathsf{discard}\ M\ \mathsf{in}\ N{:}\,B}\ \mathit{Weakening}$$

$$\frac{\Delta \triangleright M{:}\,!A \qquad \Gamma, x{:}\,!A, y{:}\,!A \triangleright N{:}\,B}{\Gamma, \Delta \triangleright \mathsf{copy}\ M\ \mathsf{as}\ x, y\ \mathsf{in}\ N{:}\,B}\ \mathit{Contraction}$$

$$\frac{\Gamma \triangleright M{:}\,!A}{\Gamma \triangleright \mathsf{derelict}(M){:}\,A}\ \mathit{Dereliction}$$

Normalization is the process of removing 'detours' from a proof in natural deduction. At the level of terms it can be seen as providing a set of reduction rules, which are known as β-rules. For **MELL** there are six β-rules which are given below.

1. $(\lambda x{:}\,A.M)\ N \rightsquigarrow_\beta M[x := N]$
2. $\mathsf{let}\ *\ \mathsf{be}\ *\ \mathsf{in}\ M \rightsquigarrow_\beta M$
3. $\mathsf{let}\ M \otimes N\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ P \rightsquigarrow_\beta P[x := M, y := N]$
4. $\mathsf{derelict}(\mathsf{promote}\ \vec{M}\ \mathsf{for}\ \vec{x}\ \mathsf{in}\ N) \rightsquigarrow_\beta N[\vec{x} := \vec{M}]$
5. $\mathsf{discard}\ (\mathsf{promote}\ \vec{M}\ \mathsf{for}\ \vec{x}\ \mathsf{in}\ N)\ \mathsf{in}\ P \rightsquigarrow_\beta \mathsf{discard}\ \vec{M}\ \mathsf{in}\ P$
6. $\mathsf{copy}\ (\mathsf{promote}\ \vec{M}\ \mathsf{for}\ \vec{x}\ \mathsf{in}\ N)\ \mathsf{as}\ y, z\ \mathsf{in}\ P \rightsquigarrow_\beta \mathsf{copy}\ \vec{M}\ \mathsf{as}\ \vec{u}, \vec{v}\ \mathsf{in}$
$P\ [y := \mathsf{promote}\ \vec{u}\ \mathsf{for}\ \vec{x}\ \mathsf{in}\ N,$
$z := \mathsf{promote}\ \vec{v}\ \mathsf{for}\ \vec{x}\ \mathsf{in}\ N]$

In addition there are other term equalities: *commuting conversions*, which arise from consideration of the subformula property, as well as those suggested by the process of cut elimination for the sequent calculus formulation.[1] For the purposes of this paper these need not be considered here. The interested reader is again referred to other work [6, 4].

[1] In fact there are other term equalities due to the interaction between our formulation of the *Promotion* rule and the fact that we are suppressing the *Exchange* rule.

2 Two Categorical Models

The fundamental idea of a categorical treatment of proof theory is that propositions should be interpreted as the objects of the category and proofs should be interpreted as morphisms. The proof rules correspond to natural transformations between appropriate hom-functors. As mentioned above, the proof theoretic setting will reveal a number of reduction rules, which can be viewed as equalities between proofs. In particular, these equalities should hold in the categorical model.

Let us fix some notation. The interpretation of a proof is represented using semantic braces, $[\![-]\!]$, making the usual simplification of using the same letter to represent a proposition as its interpretation. Given a term $\Gamma \triangleright M{:}\,A$ where $M \rightsquigarrow_\beta N$, we shall write $\Gamma \triangleright M = N{:}\,A$.

Definition 1. A category, $\mathbb{C}$, is said to be a *categorical model* of a given logic, $\mathcal{L}$, iff

1. For all proofs $\Gamma \triangleright_{\mathcal{L}} M{:}\,A$, there is a morphism $[\![M]\!]{:}\,\Gamma \to A$ in $\mathbb{C}$,
2. For all equalities $\Gamma \triangleright_{\mathcal{L}} M = N{:}\,A$ it is the case that $[\![M]\!] =_{\mathbb{C}} [\![N]\!]$ (where $=_{\mathbb{C}}$ represents equality of morphisms in the category $\mathbb{C}$).

The second condition is often referred to as 'soundness'. Given this definition we shall now consider in detail two proposals for a categorical model of Linear Logic. Firstly that proposed by Seely [14] and secondly that of Benton *et al.* [4]. First we recall Seely's definition (where for clarity we have named the natural isomorphisms relating the tensor and categorical products).

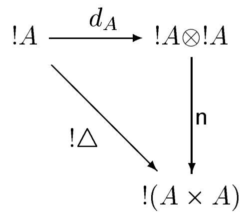
Definition 2 (Seely). A *Seely category*, $\mathbb{C}$, consists of

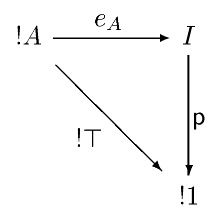
1. A symmetric monoidal closed category (SMCC) with finite products, together with a comonad $(!, \varepsilon, \delta)$, such that
2. For each object A of $\mathbb{C}$, $(!A, d_A, e_A)$ is a comonoid with respect to the tensor product,
3. There exists natural isomorphisms $\mathsf{n}{:}\,!A \otimes !B \xrightarrow{\sim} !(A \times B)$ and $\mathsf{p}{:}\,I \xrightarrow{\sim} !1$,
4. The functor ! takes the comonoid structure of the cartesian product to the comonoid structure of the tensor product.

It is instructive to consider this definition in more detail. The naturality of n amounts to the following diagram commuting for morphisms $f{:}\,A \to C$ and $g{:}\,B \to D$.

$$\begin{array}{ccc}
!A \otimes !B & \xrightarrow{\ \mathsf{n}\ } & !(A \times B) \\
{\scriptstyle !f \otimes !g}\downarrow & & \downarrow{\scriptstyle !(f \times g)} \\
!C \otimes !D & \xrightarrow[\ \mathsf{n}\ ]{} & !(C \times D)
\end{array}$$

Condition 4 (which seems to have been overlooked by Barr [2] and Troelstra [15]) amounts to requiring that the following two diagrams commute.



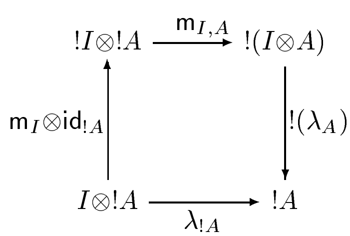


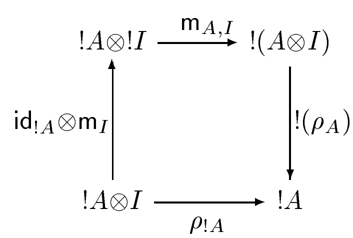
Now let us consider the model proposed by Benton *et al.* (the version given here is taken from my thesis [6] and is a slight adaptation from the original definition [4]).

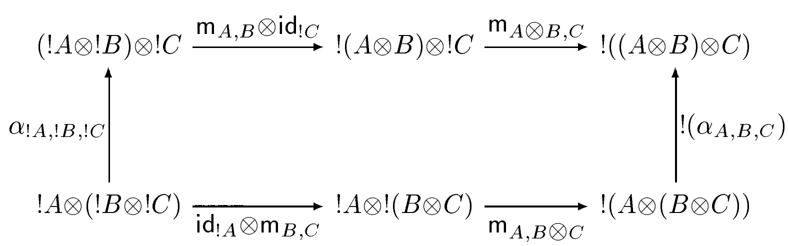
Definition 3. A *Linear category*, $\mathbb{C}$, consists of

1. A SMCC, $\mathbb{C}$, together with
2. A symmetric monoidal comonad $(!, \varepsilon, \delta, \mathsf{m}_{A,B}, \mathsf{m}_I)$ such that
   (a) For every free !-coalgebra $(!A, \delta_A)$ there are two distinguished monoidal natural transformations with components $e_A\colon !A \to I$ and $d_A\colon !A \to !A \otimes !A$[2] which form a commutative comonoid and are coalgebra morphisms,
   (b) Whenever $f\colon (!A, \delta_A) \to (!B, \delta_B)$ is a coalgebra morphism between free coalgebras, then it is also a comonoid morphism.

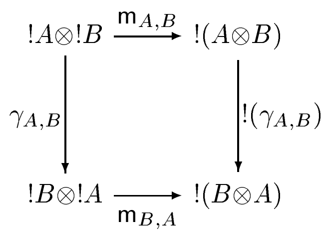
Let us consider in detail the conditions in this definition. Firstly requiring that $(!, \mathsf{m}_{A,B}, \mathsf{m}_I)$ is a symmetric monoidal functor amounts to the following diagrams commuting.



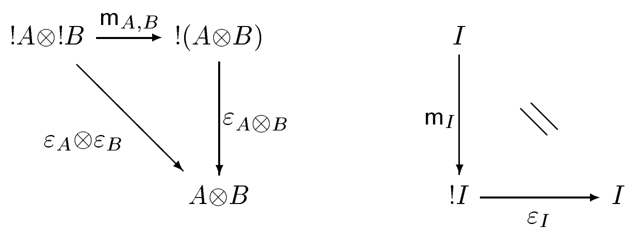




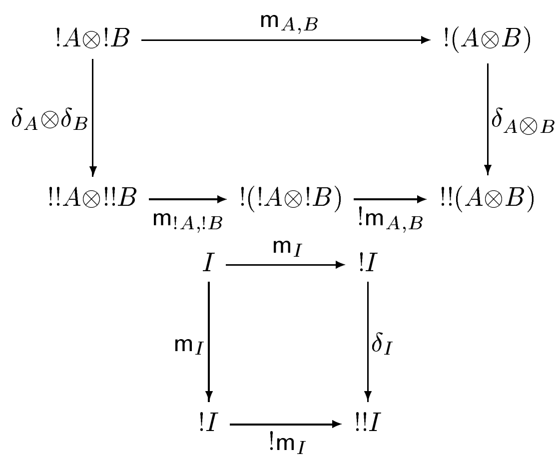
[2] This necessitates showing that $!\otimes!$ and I are monoidal functors, but this is trivial and omitted.



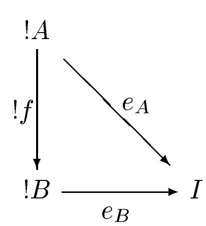
Requiring that ε is a monoidal natural transformation amounts to the following two commuting diagrams.

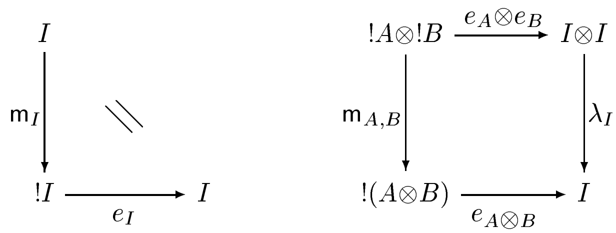


Requiring that δ is a monoidal natural transformation amounts to the following two commuting diagrams.

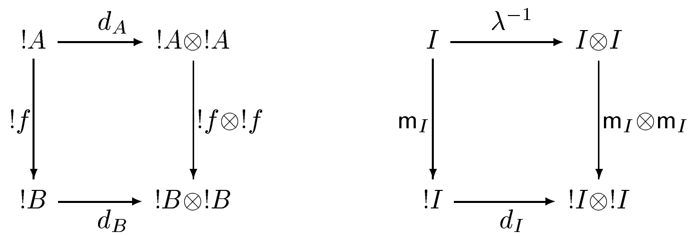


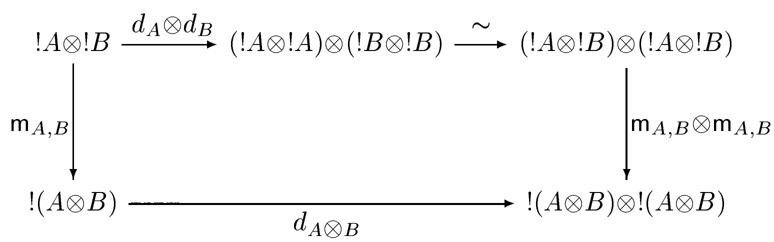
Requiring that $e_A\colon !A \to I$ is a monoidal natural transformation amounts to requiring that the following three diagrams commute, for any morphism $f\colon A \to B$.



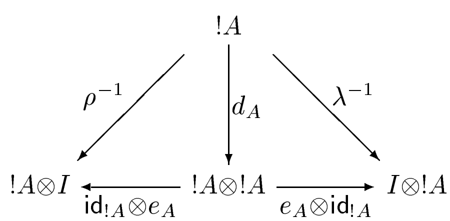


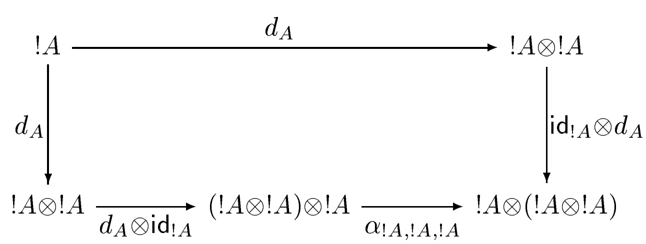
Requiring that $d_A\colon !A \to !A\otimes!A$ is a monoidal natural transformation amounts to requiring that the following three diagrams commute, for all $f\colon A \to B$.

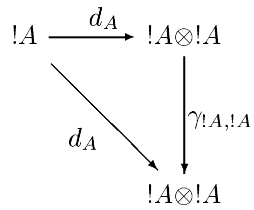




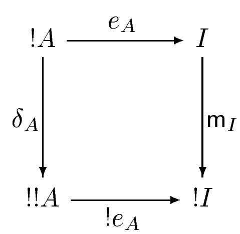
Requiring that $(!A, d_A, e_A)$ forms a commutative comonoid amounts to requiring that the following three diagrams commute.



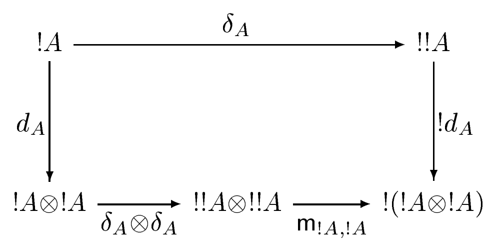




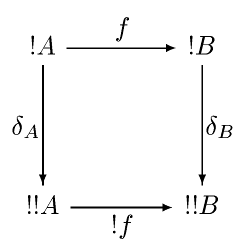
Requiring that e_A is a coalgebra morphism amounts to requiring that the following diagram commutes.



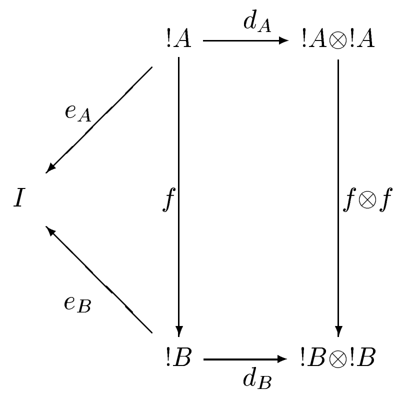
Requiring that d_A is a coalgebra morphism amounts to requiring that the following diagram commutes.



Finally all coalgebra morphisms between *free* coalgebras are also comonoid morphisms. Thus given a coalgebra morphism f, between the free coalgebras $(!A, \delta_A)$ and $(!B, \delta_B)$, i.e. which makes the following diagram commute.



Then it is also a comonoid morphism between the comonoids $(!A, e_A, d_A)$ and $(!B, e_B, d_B)$, i.e. it makes the following diagram commute.



These amount to some strong conditions on the model and some of their consequences are explored in my thesis. It is, however, reasonably straightforward to show the following.

Theorem 1. *A Linear category, $\mathbb{C}$, is a categorical model for* **MELL**.

Proof. The first condition is proved by a trivial induction on the structure of the proof $\Gamma \triangleright M : A$. The second condition is proved by checking the six β-rules from earlier.

The main difference between these two models is that a Seely category critically needs categorical products to model the exponential (!). Consider the interpretation of the *Promotion* rule. With a Seely category this is interpreted as

$$[\![\Gamma_1, \ldots, \Gamma_n \triangleright \mathsf{promote}\ M_1, \ldots, M_n\ \mathsf{for}\ x_1, \ldots, x_n\ \mathsf{in}\ N : !B]\!]$$
$$\stackrel{\text{def}}{=} [\![\Gamma_1 \triangleright M_1 : !A_1]\!] \otimes \ldots \otimes [\![\Gamma_n \triangleright M_n : !A_n]\!]; \mathsf{n}; \delta; !\mathsf{n}^{-1}; !([\![x_1 : !A_1, \ldots, x_n : !A_n \triangleright N : B]\!]).$$

With a Linear category this is interpreted as

$$[\![\Gamma_1, \ldots, \Gamma_n \triangleright \mathsf{promote}\ M_1, \ldots, M_n\ \mathsf{for}\ x_1, \ldots, x_n\ \mathsf{in}\ N : !B]\!]$$
$$\stackrel{\text{def}}{=} [\![\Gamma_1 \triangleright M_1 : !A_1]\!] \otimes \ldots \otimes [\![\Gamma_n \triangleright M_n : !A_n]\!]; \delta \otimes \ldots \otimes \delta; \mathsf{m}; !([\![x_1 : !A_1, \ldots, x_n : !A_n \triangleright N : B]\!]).$$

Let us consider whether a Seely category is a categorical model for **MELL**. Seely showed that the first requirement is satisfied.

Proposition 1 (Seely). *Given a Seely category, $\mathbb{C}$, for all proofs $\Gamma \triangleright M : A$ there is a morphism $[\![M]\!] : \Gamma \rightarrow A$ in $\mathbb{C}$.*

However the second condition is not satisfied.

Fact 1. *Given a Seely category, $\mathbb{C}$, it is* not *the case that for all term equalities $\Gamma \triangleright M = N : A$ that $[\![M]\!] =_{\mathbb{C}} [\![N]\!]$.*[3]

[3] It should be noted that the term equalities were not generally known when Seely proposed his model.

One counter-example is the sixth β-rule from earlier. In fact we only need use a simplified version where the promoted term, N, has only one free variable, i.e.

$$\begin{array}{l} \Gamma \rhd \mathsf{copy}\ (\mathsf{promote}\ M\ \mathsf{for}\ x\ \mathsf{in}\ N)\ \mathsf{as}\ y, z\ \mathsf{in}\ P \\ \quad = \mathsf{copy}\ M\ \mathsf{as}\ x', x''\ \mathsf{in}\ P[y := \mathsf{promote}\ x'\ \mathsf{for}\ x\ \mathsf{in}\ N, z := \mathsf{promote}\ x''\ \mathsf{for}\ x\ \mathsf{in}\ N] \colon C. \end{array}$$

This term equality implies the *same commuting diagram* for a Linear category as for a Seely category,

$$\begin{array}{ccccccccc} \Gamma & \xrightarrow{m} & !A & \xrightarrow{\delta} & !!A & \xrightarrow{!n} & !B & & \\ & & \Big\downarrow{\scriptstyle d} & & & & \Big\downarrow{\scriptstyle d} & & \\ & & !A\otimes !A & \xrightarrow[\delta\otimes\delta]{} & !!A\otimes !!A & \xrightarrow[!n\otimes !n]{} & !B\otimes !B & \xrightarrow[p]{} & C. \end{array} \qquad (1)$$

For a Linear category we can complete the diagram as

$$\begin{array}{ccccccccc} \Gamma & \xrightarrow{m} & !A & \xrightarrow{\delta} & !!A & \xrightarrow{!n} & !B & & \\ & & \Big\downarrow{\scriptstyle d} & & \Big\downarrow{\scriptstyle d} & & \Big\downarrow{\scriptstyle d} & & \\ & & !A\otimes !A & \xrightarrow[\delta\otimes\delta]{} & !!A\otimes !!A & \xrightarrow[!n\otimes !n]{} & !B\otimes !B & \xrightarrow[p]{} & C. \end{array}$$

The left hand square commutes by the condition that all free coalgebra morphisms are comonoid morphisms. The right hand square commutes by naturality of d. Unfortunately it is not clear how to make diagram 1 commute for a Seely category. Indeed it is straightforward to see how a term model can be constructed from Seely's original definition such that this diagram does *not* commute.

At this stage we might try adding the condition that all (free) coalgebra morphisms are comonoid morphisms to Seely's definition (and hence add extra equations to the term model). This proves still to be incomplete as we find that neither the sixth nor the fifth term equalities are modelled correctly in the cases when the promoted term, N, has zero or more than one free variable. One might be further tempted to add additional *ad-hoc* conditions to make a Seely category a model for **MELL**. However, as shown in my thesis, this is by no means simple and rather it would seem more prudent to consider a more abstract view. Rather we consider some of the motivation behind the Seely construction.

First we shall recall a construction, the dual of which (i.e. that generated by a monad) is known as the "Kleisli category" [13, Page 143].

Definition 4. Given a comonad $(!, \varepsilon, \delta)$ on a category $\mathbb{C}$, we take all the objects A in $\mathbb{C}$ and for each morphism $f\colon !A \to B$ in $\mathbb{C}$ we take a new morphism $\hat{f}\colon A \to B$. The objects and morphisms form the *co-Kleisli category* $\mathbb{C}_!$, where the composition of the morphisms $\hat{f}\colon A \to B$ and $\hat{g}\colon B \to C$ is defined as $\hat{f};\hat{g} \stackrel{\text{def}}{=} (\widehat{\delta_A; !f; g})$.

The interest in this construction is that it has strong similarities with the Girard translation [8] of Intuitionistic Logic (**IL**) into **ILL** where the intuitionistic implication is decomposed as $(A \supset B)^\circ \stackrel{\text{def}}{=} !(A^\circ) \multimap B^\circ$. In fact, as first shown by Seely [14], the co-Kleisli construction can be thought of as a categorical equivalent of the Girard translation in the following sense.

Proposition 2 (Seely). *Given a Seely category,* $\mathbb{C}$*, the co-Kleisli category,* $\mathbb{C}_!$*, is cartesian closed.*

Proof. (Sketch) Given two objects A and B their exponent is defined to be $!A \multimap B$. Then we have the following sequence of isomorphisms.

$$\begin{array}{rlr} \mathbb{C}_!(A \times B, C) & \cong \mathbb{C}(!(A \times B), C) & \text{By definition,} \\ & \cong \mathbb{C}(!A \otimes !B, C) & \text{By use of the } \mathsf{n} \text{ isomorphism,} \\ & \cong \mathbb{C}(!A, !B \multimap C) & \text{By } \mathbb{C} \text{ having a closed structure,} \\ & \cong \mathbb{C}_!(A, !B \multimap C) & \text{By definition.} \end{array}$$

We know from Kleisli's construction that we have the adjunction

$$\begin{array}{c} \mathbb{C}_! \\ G \uparrow \dashv \downarrow F \\ \mathbb{C} \end{array}$$

where G is the functor defined by $g: A \to B \mapsto \widehat{(\varepsilon; g)}$ and F is the functor defined by $\hat{f}: A \to B \mapsto \delta_A; !f$.

Seely's model arises from at least the desire to make the co-Kleisli category a cartesian closed category (CCC), which is achieved by including the n and p natural isomorphisms. This means that there is an adjunction between a SMCC ($\mathbb{C}$) and a CCC ($\mathbb{C}_!$). As a CCC is trivially a SMCC, there is then an adjunction between two SMCCs. We might expect that this is a *monoidal adjunction*.

Definition 5. An adjunction $\langle F, G, \eta, \epsilon \rangle : \mathbb{C} \rightharpoonup \mathbb{D}$ is said to be a *monoidal* adjunction when F and G are monoidal functors and η and ϵ are monoidal natural transformations.

We now state a new definition for a Seely-style category and then investigate some of its properties.

Definition 6. A *new-Seely category*, $\mathbb{C}$, consists of

1. a SMCC, $\mathbb{C}$, with finite products, together with
2. a comonad, $(!, \varepsilon, \delta)$, and
3. two natural isomorphisms, $\mathsf{n}: !A \otimes !B \xrightarrow{\sim} !(A \times B)$ and $\mathsf{p}: I \xrightarrow{\sim} !1$

such that the adjunction, $\langle F, G, \eta, \varepsilon \rangle$, between $\mathbb{C}$ and $\mathbb{C}_!$ is a *monoidal* adjunction.

Assuming that F is monoidal gives us the morphism and natural transformation

$$\begin{aligned}&\mathsf{m}_I\colon I \to F1,\\&\mathsf{m}_{A,B}\colon FA\otimes FB \to F(A\times B).\end{aligned}$$

Assuming that G is monoidal gives us the morphism and natural transformation

$$\begin{aligned}&\mathsf{m}'_1\colon 1 \to GI,\\&\mathsf{m}'_{A,B}\colon GA\times GB \to G(A\otimes B).\end{aligned}$$

By assumption ε and η are monoidal natural transformations.

It is easy to see that m_I is Seely's morphism p and $\mathsf{m}_{A,B}$ is Seely's natural transformation n. In fact, we can define their inverses

$$\begin{aligned}&\mathsf{m}_I^{-1} \stackrel{\text{def}}{=} F\mathsf{m}'_1;\varepsilon_I\colon F1 \to I,\\&\mathsf{m}_{A,B}^{-1} \stackrel{\text{def}}{=} F(\eta_A\times\eta_B);F\mathsf{m}'_{FA,FB};\varepsilon_{FA\otimes FB}\colon F(A\times B)\to FA\otimes FB.\end{aligned}$$

Hence the monoidal adjunction itself provides the isomorphisms $!A\otimes !B\cong !(A\times B)$ and $I\cong !1$. As the co-Kleisli category is a CCC it has a trivial commutative comonoid structure, $(A,\triangle,\top)$, on all objects A. We can use this and the natural transformations arising from the monoidal adjunction to define a comonoid structure, $(F(A),d,e)$, on the objects of $\mathbb{C}$, with the structure maps defined as

$$\begin{aligned}d &\stackrel{\text{def}}{=} F(\triangle);\mathsf{m}_{A,A}^{-1}\colon F(A)\to F(A)\otimes F(A),\\ e &\stackrel{\text{def}}{=} F(\top);\mathsf{m}_I^{-1}\colon F(A)\to I.\end{aligned}$$

It is easy to see that these definitions amount to condition 4 of Seely's original definition. Thus there is at least as much structure as in Seely's original definition but with the extra structure of the monoidal adjunction. Some consequences of this adjunction are given in the following lemma.

Lemma 1. *Given a new-Seely category, $\mathbb{C}$, the following facts hold:*

1. *The induced comonad $(FG,F\eta_G,\varepsilon)$ on $\mathbb{C}$ is a* monoidal comonad $(FG,F\eta_G,\varepsilon,m_{A,B},m_I)$.
2. *The comonoid morphisms $e\colon FG(A)\to I$ and $d\colon FG(A)\to FG(A)\otimes FG(A)$ are monoidal natural transformations.*
3. *The comonoid morphisms $e\colon FG(A)\to I$ and $d\colon FG(A)\to FG(A)\otimes FG(A)$ are coalgebra morphisms.*
4. *If $f\colon(FG(A),F\eta_{GA})\to(FG(B),F\eta_{GB})$ is a coalgebra morphism then it is also a comonoid morphism.*

Proof. For part 1 we take the definitions

$$\begin{aligned}m_I &\stackrel{\text{def}}{=} \mathsf{m}_I;F\mathsf{m}'_1\colon I\to FG(I),\\ m_{A,B} &\stackrel{\text{def}}{=} \mathsf{m}_{GA,GB};F\mathsf{m}'_{A,B}\colon FG(A)\otimes FG(A)\to FG(A\otimes B).\end{aligned}$$

The rest of the lemma holds by construction.

Corollary 1. *Every new-Seely category is a Linear category.*

(It is clear that the converse is not true, as the Linear category need not have finite products.) We can hence show that a new-Seely category is a sound model for the **MELL**.

Theorem 2. *A new-Seely category, $\mathbb{C}$, is a categorical model for* **MELL**

3 Including the additives

Now we shall consider the whole of **ILL** by adding the additive connectives to **MELL**. Logically these are given by the following sequent calculus rules (we shall ignore the additive units).

$$\frac{\Gamma, A \vdash C}{\Gamma, A \times B \vdash C}\,(\times_{\mathcal{L}-1}) \qquad \frac{\Gamma, B \vdash C}{\Gamma, A \times B \vdash C}\,(\times_{\mathcal{L}-2})$$

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \times B}\,(\times_{\mathcal{R}})$$

$$\frac{\Gamma, A \vdash C \qquad \Gamma, B \vdash C}{\Gamma, A \oplus B \vdash C}\,(\oplus_{\mathcal{L}})$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \oplus B}\,(\oplus_{\mathcal{R}-1}) \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \oplus B}\,(\oplus_{\mathcal{R}-2})$$

There are a number of ways of formulating the additives in a natural deduction system which are discussed in my thesis. However, for now we shall simply take the term assignment system which is familiar from that of the λ-calculus. The term assignment rules as well as the β-rules for the additives are given below.

$$\frac{\Gamma \triangleright M: A \qquad \Gamma \triangleright N: B}{\Gamma \triangleright \langle M, N\rangle: A \times B}\,(\times_{\mathcal{I}})$$

$$\frac{\Gamma \triangleright M: A \times B}{\Gamma \triangleright \mathsf{fst}(M): A}\,(\times_{\mathcal{E}-1}) \qquad \frac{\Gamma \triangleright M: A \times B}{\Gamma \triangleright \mathsf{snd}(M): B}\,(\times_{\mathcal{E}-2})$$

$$\frac{\Gamma \triangleright M: A}{\Gamma \triangleright \mathsf{inl}(M): A \oplus B}\,(\oplus_{\mathcal{I}-1}) \qquad \frac{\Gamma \triangleright M: B}{\Gamma \triangleright \mathsf{inr}(M): A \oplus B}\,(\oplus_{\mathcal{I}-2})$$

$$\frac{\Delta \triangleright M: A \oplus B \qquad \Gamma, x: A \triangleright N: C \qquad \Gamma, y: B \triangleright P: C}{\Gamma, \Delta \triangleright \mathsf{case}\ M\ \mathsf{of}\ \mathsf{inl}(x) \rightarrow N \parallel \mathsf{inr}(y) \rightarrow P: C}\,(\oplus_{\mathcal{E}})$$

$$\begin{aligned}
\mathsf{fst}(\langle M, N\rangle) &\leadsto_\beta M\\
\mathsf{snd}(\langle M, N\rangle) &\leadsto_\beta N\\
\mathsf{case}\ (\mathsf{inl}(M))\ \mathsf{of}\ \mathsf{inl}(x)\ \to\ N \parallel \mathsf{inr}(y)\ \to\ P &\leadsto_\beta N[x := M]\\
\mathsf{case}\ (\mathsf{inr}(M))\ \mathsf{of}\ \mathsf{inl}(x)\ \to\ N \parallel \mathsf{inr}(y)\ \to\ P &\leadsto_\beta P[y := M]
\end{aligned}$$

To model these additive connectives we shall add finite products and coproducts to a Linear category and finite coproducts to a new-Seely category. As might be expected both models are sound.

Theorem 3. *Both a new-Seely category with finite coproducts and a Linear category with finite products and coproducts, are models for* **ILL**.

Somewhat surprisingly, we find that the so-called Seely isomorphisms (n and p) exist in a Linear category with products.

Lemma 2. *Given a Linear category with finite products we can define the natural isomorphisms*

$$\begin{aligned}
n &\stackrel{\text{def}}{=} \delta\otimes\delta; \mathsf{m}_{!A,!B}; !(\triangle); !((\mathsf{id}\otimes e_B) \times (e_A\otimes\mathsf{id})); !(\rho \times \lambda); !(\varepsilon \times \varepsilon) \colon !A\otimes !B \to !(A \times B),\\
n^{-1} &\stackrel{\text{def}}{=} d_{A\times B}; !\mathsf{fst}\otimes !\mathsf{snd}\colon !(A \times B) \to !A\otimes !B,\\
p &\stackrel{\text{def}}{=} \mathsf{m}_I; !\top\colon I \to !1,\\
p^{-1} &\stackrel{\text{def}}{=} e_1\colon !1 \to I.
\end{aligned}$$

Thus the co-Kleisli category associated with a Linear category is also a CCC. Given our earlier calculations we might consider the adjunction between a Linear category and its co-Kleisli category, where we find the following holds.

Lemma 3. *The adjunction between a Linear category, $\mathbb{C}$, with finite products and its co-Kleisli category, $\mathbb{C}_!$, is a* monoidal *adjunction.*

Thus when considering the complete intuitionistic fragment, the new-Seely and Linear categories are equivalent. It is easy to check that common models such as coherent spaces, dI-domains and pointed cpos and strict maps are all examples of new-Seely/Linear categories.

An interesting question is whether the co-Kleisli category $\mathbb{C}_!$ has an induced coproduct structure given a coproduct structure on $\mathbb{C}$. Seely [14] showed that $\mathbb{C}_!$ does *not* have a coproduct structure, but in fact it is possible to identify a *weak* coproduct structure. We use the following well-known fact about the co-Kleisli category [12, Corollary 6.9].

Fact 2. *The co-Kleisli category of a comonad is equivalent to the full subcategory of the category of coalgebras consisting of the free coalgebras.*

Lemma 4. *Given two free coalgebras $(!A, \delta_A)$ and $(!B, \delta_B)$, we define their coproduct to be $(!(!A\oplus !B), \delta_{!A\oplus !B})$. We define the injection morphisms to be inl $\stackrel{\text{def}}{=}$*

$\delta_A; !\mathsf{inl}\colon !A \to !(!A \oplus !B)$ *and* $inr \stackrel{\text{def}}{=} \delta_B; !\mathsf{inr}\colon !B \to !(!A \oplus !B)$*, which are (free) coalgebra morphisms. Given two (free) coalgebra morphisms* $f\colon !A \to !C$ *and* $g\colon !B \to !C$*, then the morphism* $(![f,g]; !\varepsilon_C)\colon !(!A \oplus !B) \to !C$ *is a (free) coalgebra morphism and makes a coproduct diagram commute.*

So far we have followed others [14, 11] and only considered whether the co-Kleisli category $\mathbb{C}_!$ generated by the comonad is cartesian closed. It is should be noted that alternatively one can consider the full Eilenberg-Moore category of coalgebras ($\mathbb{C}^!$) instead. In other work [4, 6], various subcategories of $\mathbb{C}^!$ are shown to be cartesian closed. An important feature of these (sub)categories is that the underlying category $\mathbb{C}$ need *not* necessarily have products, in contrast to the situation for $\mathbb{C}_!$. The interested reader is referred to these other works.

4 Conclusions

In this paper we have considered the definition of a categorical model for **ILL**. Surprisingly, Seely's now standard definition [14] was shown to be unsound, in that it does not model all equal proofs with equal morphisms. A model given in our earlier work [4] was shown to be sound. We have also considered a method for improving Seely's original definition so as to be sound. In fact both (sound) models turn out to be equivalent. It is worth pointing out that these models are sound with respect to the equalities arising from the commuting conversions.

Lafont [11] also proposed a categorical model for **ILL**, which amounts to requiring an adjunction between a SMCC and a category of commutative comonoids. In my thesis [6] it is shown that this model is a categorical model of **ILL** by demonstrating that every Lafont category is a Linear category.

In Lemma 1 it was proved that a monoidal adjunction between a particular SMCC (a new-Seely category) and CCC (its co-Kleisli category) yielded the structure of a Linear category. Lemma 3 shows that a Linear category also has the structure of a monoidal adjunction between it (a SMCC) and its associated co-Kleisli category (a CCC). Thus the notion of a Linear category is in some senses equivalent to the existence of a monoidal adjunction between a SMCC and a CCC. This observation has been used by Benton [3] to derive the syntax of a mixed linear and non-linear term calculus.

Categorically, most models proposed for Classical Linear Logic (**CLL**) are extensions of Seely's model for **ILL** to $\star$-autonomous categories [14, 2]. Thus the problems identified with Seely's model in this paper apply to these models. Extending a Linear category with a dualizing object gives a (sound) model of **CLL**, although the categorical import of this construction is work in progress.

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